Robustly quasiconvex function

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Convex functions

1. All lower level sets are convex.
2. Each local minimum is a global minimum.
3. Each stationary point is a global minimizer.
Generalised Convexity

**Definition**

A function $f : X \to \overline{\mathbb{R}}$, with a convex $\text{dom}f$, is called **quasiconvex** if for all $x, y \in \text{dom}f$, and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}.$$
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**Definition**
$f$ is called **explicitly quasiconvex** if it is quasiconvex and for all $\lambda \in (0, 1)$

$$f(\lambda x + (1 - \lambda)y) < \max\{f(x), f(y)\}, \quad \text{with } f(x) \neq f(y).$$
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A function \( f : X \to \mathbb{R} \), with a convex \( \text{dom} f \), is called **quasiconvex** if for all \( x, y \in \text{dom} f \), and \( \lambda \in [0, 1] \) we have
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\]

**Example**
1. \( f_1 : \mathbb{R} \to \mathbb{R}; \ f_1(x) = 0, \ x \neq 0; f_1(0) = 1. \)
2. \( f_2 : \mathbb{R} \to \mathbb{R}; \ f_2(x) = 1, \ x \neq 0; f_2(0) = 0. \)
3. Convex functions are quasiconvex, and explicitly quasiconvex.
4. \( f(x) = x^3 \) are quasiconvex, and explicitly quasiconvex, but not convex.
Definition

A Gâteaux differentiable function $f : X \rightarrow \mathbb{R}$, with a convex $\text{dom} f$, is called \textit{pseudoconvex} if for $x, y \in \text{dom} f$,

$$f(x) < f(y) \Rightarrow \langle \nabla f(y), x - y \rangle < 0.$$
Pseudoconvex function

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**Example**

$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = \int_{-1}^{x} t \sin(1/t) dt + x^2$. $f$ is pseudoconvex but not stable.
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**Example**

$f : \mathbb{R} \to \mathbb{R}, f(x) = \int_{-1}^{x} t \sin(1/t)dt + x^2$. $f$ is pseudoconvex but not stable.
Quasiconvexity satisfies (1), not (2), (3), and explicitly quasiconvexity satisfies (1), (2), not (3).

Pseudoconvexity satisfies (3).

These three properties are not stable w/r to three generalized convex properties.
Robust quasiconvexity

$X$–Banach space.

**Definition**

For $\alpha > 0$, a function $f : X \to \overline{\mathbb{R}}$ is called $\alpha$–robustly quasiconvex if, for every $v^* \in \alpha B^*$, the function $f_{v^*} : v \mapsto f(x) + \langle v^*, x \rangle$ is quasiconvex.
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A function is called $s$–quasiconvex if there is $\alpha > 0$ such that $f$ is $\alpha$–robustly quasiconvex.
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A function is call s–quasiconvex if there is $\alpha > 0$ such that $f$ is $\alpha$–robustly quasiconvex.

1. s–quasiconvex functions satisfy (1), (2), (3).
2. Consequently, these three properties are stable under a small linear perturbation.
3. A lower semicontinuous $\alpha$–robustly quasiconvex function are not necessary continuous in its int dom.
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2. Consequently, these three properties are stable under a small linear perturbation.

3. A lower semicontinuous $\alpha$–robustly quasiconvex function are not necessary continuous in its int dom.

**Theorem (J.-P. Crouzeix, 1977)**

A function $f : X \to \overline{\mathbb{R}}$ is convex provided all its linear perturbations $f + x^*$, $x^* \in X^*$ are quasiconvex.
Convex subdifferential set of $\varphi$ at $\bar{x}$

$$\partial \varphi(\bar{x}) := \{ x^* \in X^* : \varphi(x) - \varphi(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle, \forall x \in X \}.$$
Dual Characterization: subdifferentials

$X$— normed vector space, $\varphi : X \rightarrow \bar{\mathbb{R}}$, $\bar{x} \in X$

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$$\partial^F \varphi(\bar{x}) := \left\{ x^* \in X^* : \liminf_{x \rightarrow \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\| x - \bar{x} \|} \geq 0 \right\}.$$
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**Theorem**

1. **$\varphi$ is convex, then**
   $$\partial^F \varphi(x) = \partial \varphi(x), \quad x \in X.$$

2. **$x^* \in \partial^F \varphi(x)$ if and only if there exists a function $g : X \to \mathbb{R}$ such that**
   1. $g(x) = \varphi(x)$, and $g(y) \leq \varphi(y)$ for all $y \in X$.
   2. $g$ is Fréchet differentiable at $x$, and $\nabla g(x) = x^*$.

3. **$\bar{x}$ is a local minimum of $\varphi$, then**
   $$0 \in \partial^F \varphi(\bar{x}).$$
Examples

\[ \varphi(x) = \begin{cases} \max\{0, x \sin(1/x)\} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases} \]
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\[ \partial^F \varphi(0) = \{0\}; \text{ but } \varphi \text{ is not differentiable at } 0. \]
Examples

$X = \mathbb{R}^2$ and $\varphi : X \to \mathbb{R}$

$$\varphi(x) = |x_1| - |x_2|, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$
Examples

$X = \mathbb{R}^2$ and $\varphi : X \to \mathbb{R}$

$$\varphi(x) = |x_1| - |x_2|, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$
An operator $F : X \rightrightarrows X^*$ is called **monotone** if

$$\langle v^* - u^*, v - u \rangle \geq 0,$$

for all $u, v \in X$ and $u^* \in F(u), v^* \in F(v)$.

$F$ is **maximal monotone** if there is no monotone operator that properly contains it.
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Theorem (Rockafellar (1970))

If $\varphi$ is convex, then $\partial F \varphi$ is maximal monotone.
A Banach space $X$ is said to be **Asplund** if every continuous convex function defined on a nonempty open convex subset $D$ of $X$ is Fréchet differentiable at each point of some dense set $G_δ$ subset of $D$. 

**Theorem**

If $X$ is Asplund, $f : X → \overline{\mathbb{R}}$, then $f$ is convex if and only if $\partial Ff$ is monotone.

$X$ is NOT Asplund if and only if there is a proper l.s.c function $ϕ : X → \overline{\mathbb{X}}$ with $\partial Fϕ$ is monotone, but not convex.

**Lemma (Lemma of three points)**

Let $ϕ : X → \mathbb{R}$ be a proper, l.s.c on Asplund space $X$. Let $u, v, w ∈ X$ with $v ∈ [u, w]$, $ϕ(v) > ϕ(u)$. Then, there exist $c ∈ [u, v)$ and two sequences $x_k \rightarrow c$, and $x_k^* ∈ ∂ Fϕ(x_k)$ such that $⟨ x_k^*, w − x_k ⟩ > 0 \forall k ∈ \mathbb{N}$.

(Ausel, 1988) $∂^−$ smooth renorm space.
Asplund space

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**Lemma (Lemma of three points)**

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$$\langle x_k^*, w - x_k \rangle > 0 \quad \forall k \in \mathbb{N}.$$  

(Ausel, 1988) $\partial$—smooth renorm space.
Quasiconvex function-Quasimonotone operator

**Definition**

An operator $F : X \rightarrow X^*$ is quasimonotone if

$$\langle u^*, v - u \rangle > 0 \Rightarrow \langle v^*, v - u \rangle \geq 0$$

for all $u, v \in X$ and $u^* \in F(u), v^* \in F(v)$.

Monotone $\implies$ Quasimonotone.
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**Theorem**

Let $\varphi : X \to \overline{\mathbb{R}}$ be a proper, l.s.c function on a Banach space $X$. Consider the following statements

(a) $\varphi$ is quasiconvex;
(b) $\varphi(y) \leq \varphi(x) \implies \langle x^*, y - x \rangle \leq 0 \quad \forall x^* \in \partial F \varphi(x)$.
(c) $\partial^F \varphi$ is quasimonotone.

Then, (a)$\Rightarrow$(b) $\iff$ (c).
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An operator \( F : X \rightarrow X^* \) is **quasimonotone** if

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(c) \( \partial^F \varphi \) is quasimonotone.

Then, (a)\( \Rightarrow \)(b) \( \iff \) (c). Moreover, if \( X \) is an Asplund space then (c)\( \Rightarrow \)(a).
Definition

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(c) \( \partial^F \varphi \) is quasimonotone.

Then, (a)\( \Rightarrow \) (b) \( \iff \) (c). Moreover, if \( X \) is an Asplund space then (c)\( \Rightarrow \) (a).

If \( X \) is NOT Asplund, we can always find a proper l.s.c function that satisfies (c), (b); but not (a).
Let \( \varphi : X \to \overline{\mathbb{R}} \) be a proper, l.s.c function on a Banach space \( X \), and \( \alpha > 0 \). Consider the following statements

(a) \( \varphi \) is \( \alpha \)-robustly quasiconvex;

(b) For every \( x, y \in X \)

\[
\varphi(y) \leq \varphi(x) \implies \langle x^*, y - x \rangle \leq -\min \{ \alpha \|y - x\|, \varphi(x) - \varphi(y) \}, \quad \forall x^* \in \partial^F \varphi(x).
\]

Then, (a) \( \implies \) (b).
Theorem

Let \( \varphi : X \rightarrow \overline{\mathbb{R}} \) be a proper, l.s.c function on a Banach space \( X \), and \( \alpha > 0 \). Consider the following statements

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\( \forall x^* \in \partial^F \varphi(x) \).

Then, (a) \( \Rightarrow \) (b). Moreover, if \( X \) is an Asplund space then (b) \( \Rightarrow \) (a).

Theorem

Let \( \varphi : X \rightarrow \overline{\mathbb{R}} \) be proper, l.s.c on an Asplund space \( X \) and \( \alpha \geq 0 \). Then, \( \varphi \) is \( \alpha \)-robustly quasiconvex if and only if for any \( x, y \in X \) and \( x^* \in \partial^F \varphi(x), y^* \in \partial^F \varphi(y) \), we have

\[
\min \{ \langle x^*, y - x \rangle, \langle y^*, x - y \rangle \} > -\alpha \|y - x\| \implies \langle x^* - y^*, x - y \rangle \geq 0.
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2. Each local minimum is a global minimum.
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THANK YOU!