A unified convergence analysis framework for numerical approximations of diffusion equations

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RMIT
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Joint work with many collaborators...
How are you with combinatorics?

If we are to analyse the convergence of each numerical method for each model:

<table>
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Each line = one analysis to perform... add to that all the benchmarking!
How about going through some sort of generic framework?

**Objective**: the framework identifies a few key properties that all methods satisfy, and that are sufficient for all convergence analyses.
Plan

1. Model PDE and weak formulation
2. Towards the gradient discretisation method
3. The GDM
   - General framework
   - Measures of accuracy, error estimates
   - $3+1+1=5$ properties for convergence
4. Examples of GDMs
5. Miscellanea
   - Proving (P1)–(P4)
   - Using GDM for non-linear models
   - Novel result from the GDM: super-convergence of TPFA finite volumes
6. Towards numerical analysis for calculus of variations (?)
\[ \begin{cases} -\Delta \bar{u} = f & \text{in } \Omega \\ \bar{u} = 0 & \text{on } \partial \Omega \end{cases} \]

- \( \Omega \) open bounded in \( \mathbb{R}^d \),
- \( f : \Omega \to \mathbb{R} \).

▶ Prototypal of diffusion models, e.g. heat equation, flows in porous media, etc.
Weak formulation of the Laplace equation

\[
\begin{align*}
-\Delta \overline{u} &= f \quad \text{in} \; \Omega \\
\overline{u} &= 0 \quad \text{on} \; \partial \Omega
\end{align*}
\]

Divergence formula (integration-by-parts): for \( F : \Omega \to \mathbb{R}^d \) and \( \nu : \Omega \to \mathbb{R} \), if \( n \) is the outer normal to \( \partial \Omega \),

\[
\int_{\Omega} \nu \text{div}(F) = \int_{\partial \Omega} \nu F \cdot n - \int_{\Omega} F \cdot \nabla \nu
\]

where \( \nabla \nu = (\partial_1 \nu, \ldots, \partial_d \nu) \).
Weak formulation of the Laplace equation

\[ \begin{cases} 
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**Divergence formula (integration-by-parts):** for \( F : \Omega \to \mathbb{R}^d \) and \( \nu : \Omega \to \mathbb{R} \), if \( n \) is the outer normal to \( \partial \Omega \),

\[ \int_{\Omega} \nu \text{div}(F) = \int_{\partial \Omega} \nu F \cdot n - \int_{\Omega} F \cdot \nabla \nu \]

where \( \nabla \nu = (\partial_1 \nu, \ldots, \partial_d \nu) \).

**For Laplace equation:** since \( \Delta \bar{u} = \text{div}(\nabla \bar{u}) \), multiply equation by \( \bar{\nu} \) such that \( \bar{\nu} = 0 \) on \( \partial \Omega \) and integrate by parts:

\[ \int_{\Omega} \nabla \bar{u} \cdot \nabla \bar{\nu} = \int_{\Omega} f \bar{\nu}. \]
Weak formulation of the Laplace equation

\[ \overline{u} = 0 \text{ on } \partial \Omega \text{ such that, for all } \overline{v} = 0 \text{ on } \partial \Omega, \]

\[ \int_{\Omega} \nabla \overline{u} \cdot \nabla \overline{v} = \int_{\Omega} f \overline{v}. \]

**Riesz representation theorem:** If \( \langle \cdot, \cdot \rangle \) is an inner product in \( \mathbb{R}^N \), for any \( \ell : \mathbb{R}^N \rightarrow \mathbb{R} \) linear, there is a unique solution to

Find \( X \in \mathbb{R}^N \) s.t., for all \( Y \in \mathbb{R}^N \), \( \langle X, Y \rangle = \ell(Y) \).
Weak formulation of the Laplace equation

\( \bar{u} = 0 \) on \( \partial \Omega \) such that, for all \( \bar{v} = 0 \) on \( \partial \Omega \),

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Find \( X \in \mathbb{R}^N \) s.t., for all \( Y \in \mathbb{R}^N \), \( \langle X, Y \rangle = \ell(Y) \).

“Application” to Laplace: define

\[
\langle \bar{u}, \bar{v} \rangle = \int_{\Omega} \nabla \bar{u} \cdot \nabla \bar{v}, \quad \ell(\bar{v}) = \int_{\Omega} f \bar{v}.
\]

\( \bar{u}, \bar{v} \) are not vectors in \( \mathbb{R}^N \), but functions (infinite dimensional space!).

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Weak formulation

**Hilbert space**: Riesz representation theorem is also valid in *Hilbert* spaces.

\[ H^1_0(\Omega) = \left\{ \overline{v} : \Omega \to \mathbb{R} : \int_{\Omega} |v|^2 < +\infty , \int_{\Omega} |\nabla v|^2 < +\infty , \right. \]
\[ \left. \overline{v} = 0 \text{ on } \partial\Omega \right\} \]

with inner product \( \langle \overline{u}, \overline{v} \rangle = \int_{\Omega} \nabla \overline{u} \cdot \nabla \overline{v} \).
Weak formulation

**Hilbert space**: Riesz representation theorem is also valid in *Hilbert* spaces.

\[ H^1_0(\Omega) = \left\{ \nabla : \Omega \rightarrow \mathbb{R} : \int_{\Omega} |v|^2 < +\infty, \int_{\Omega} |\nabla v|^2 < +\infty, \quad v = 0 \text{ on } \partial \Omega \right\} \]

with inner product \( \langle u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v \).

**Weak formulation**: Find \( \bar{u} \in H^1_0(\Omega) \) s.t., for all \( \bar{v} \in H^1_0(\Omega) \), \( \int_{\Omega} \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v} \).

▶ Existence and uniqueness.
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Galerkin approximation

Find \( \bar{u} \in H^1_0(\Omega) \) s.t., for all \( \bar{v} \in H^1_0(\Omega) \), \( \int_{\Omega} \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v} \).

**Computational issue:** \( H^1_0(\Omega) \) is an infinite-dimensional space.
\( \leadsto \) Cannot be understood/manipulated by computer.
Find $\bar{u} \in H^1_0(\Omega)$ s.t., for all $\bar{v} \in H^1_0(\Omega)$, $\int_{\Omega} \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v}$.

**Computational issue:** $H^1_0(\Omega)$ is an infinite-dimensional space. 

~$\Rightarrow$ Cannot be understood/manipulated by computer.

**Easy solution:** take $E \subset H^1_0(\Omega)$ finite-dimensional subspace and write

Find $u_E \in E$ s.t., for all $v_E \in E$, $\int_{\Omega} \nabla u_E \cdot \nabla v_E = \int_{\Omega} f v_E$. 
Example: $P_1$ finite elements

**Space $E$:** cut $\Omega$ into triangles and consider continuous functions that are affine in each triangle (and zero on the boundary).

- Described by values at the vertices (finite number of such values).
Non-conforming \( P_1 \)

Non-conforming approximation: \( E \notsubset H_0^1(\Omega) \).

\[ \mapsto \text{Define } \nabla v_E \text{ if } v_E \in E? \]
Non-conforming approximation: $E \not\subset H^1_0(\Omega)$.

Define $\nabla v_E$ if $v_E \in E$?

Example: non-conforming $P_1$ on a triangular mesh:

$$E = E_h = \{ v_h : \Omega \to \mathbb{R} : v_h \text{ piecewise linear, continuous at edge midpoints, zero at boundary edge midpoints} \}$$
Non-conforming \( P_1 \)

Non-conforming approximation: \( E \not\subset H^1_0(\Omega) \).

\[ \leadsto \text{Define } \nabla v_E \text{ if } v_E \in E? \]

Example: non-conforming \( P_1 \) on a triangular mesh:
- \( \nabla u_h \) replaced with broken gradient \( \nabla_h u_h \), computed cell-wise.

Find \( u_h \in E_h \) s.t., for all \( v_h \in E_h \),

\[ \int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h. \]
Mass-lumping

- Non-linear diffusion-reaction \( (e.g., \text{from Euler time-stepping on Richards' equation}) \): with \( \beta(s)s \geq 0 \),

\[-\Delta \bar{u} + \beta(\bar{u}) = f.\]

- Weak formulation:

\[
\bar{u} \in H^1_0(\Omega), \forall v \in H^1_0(\Omega), \quad \int_{\Omega} \nabla \bar{u} \cdot \nabla v + \int_{\Omega} \beta(\bar{u})v = \int_{\Omega} f v.
\]
Mass-lumping

With $\beta(s)s \geq 0$,

$$
\overline{u} \in H^1_0(\Omega), \forall v \in H^1_0(\Omega), \quad \int_\Omega \nabla \overline{u} \cdot \nabla v + \int_\Omega \beta(\overline{u})v = \int_\Omega f v.
$$

**Numerical approximation**: e.g., non-conforming $P_1$ finite elements: find $u_h \in E_h$ s.t., $\forall v_h \in E_h$,

$$
\int_\Omega \nabla_h u_h \cdot \nabla_h v_h + \left\{ \begin{array}{l}
\int_\Omega \beta(u_h)v_h \\
\int_\Omega [\beta(u_{\text{nodal}})]_hv_h
\end{array} \right\} = \int_\Omega f v_h.
$$
Mass-lumping

With $\beta(s) \geq 0$,

$$
\overline{u} \in H^1_0(\Omega), \quad \forall v \in H^1_0(\Omega), \quad \int_\Omega \nabla \overline{u} \cdot \nabla v + \int_\Omega \beta(\overline{u}) v = \int_\Omega f v.
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$$
\int_\Omega \nabla_h u_h \cdot \nabla_h v_h + \left\{ \begin{array}{l}
\int_\Omega \beta(u_h) v_h \\
\int_\Omega \left[ \beta(u_{\text{nodal}}) \right]_{h} v_h \\
\end{array} \right. = \int_\Omega f v_h.
$$

**Issues**:

- No exact quadrature for $\int_\Omega \beta(u_h) v_h$.
- No certainty that $\int_\Omega \left[ \beta(u_{\text{nodal}}) \right]_{h} u_h \geq 0$. 

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**Mass-lumping**

**Idea:** replace functions by piecewise-constant reconstructions:

Find $u_h \in E_h$ s.t., for all $v_h \in E_h$,

$$
\int_{\Omega} \nabla_h u_h \cdot \nabla_h v_h + \int_{\Omega} \beta(\Pi_h u_h) \Pi_h v_h = \int_{\Omega} f \Pi_h v_h.
$$
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Step 1: select the discrete space and operators

Gradient discretisation: \( D = (X_D,0, \Pi_D, \nabla_D) \)

- \( X_D,0 \) finite dimensional space (encodes the discrete unknowns of the scheme, and accounts for the BCs).

  - **Conforming \( \mathbb{P}_1 \):** with \( V = \) vertices of the mesh,
    \[
    X_D,0 = \{ u = (u_v)_{v \in V} : u_v = 0 \text{ if } v \in \partial \Omega \}.
    \]

  - **Non-conforming \( \mathbb{P}_1 \):** with \( F = \) faces of the mesh,
    \[
    X_D,0 = \{ u = (u_\sigma)_{\sigma \in F} : u_\sigma = 0 \text{ if } \sigma \subset \partial \Omega \}.
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Step 1: select the discrete space and operators

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    \]

- \( v \in X_{D,0} \mapsto \Pi_D v \) linear.
  \( \Pi_D v : \Omega \to \mathbb{R} \) “reconstructed function”.
Step 1: select the discrete space and operators

Gradient discretisation: $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$

- $X_{\mathcal{D},0}$ finite dimensional space (encodes the discrete unknowns of the scheme, and accounts for the BCs).
  - **Conforming $\mathbb{P}_1$:** with $\mathcal{V} = \text{vertices of the mesh}$,
    \[ X_{\mathcal{D},0} = \{ u = (u_v)_{v \in \mathcal{V}} : u_v = 0 \text{ if } v \in \partial \Omega \}. \]
  - **Non-conforming $\mathbb{P}_1$:** with $\mathcal{F} = \text{faces of the mesh}$,
    \[ X_{\mathcal{D},0} = \{ u = (u_\sigma)_{\sigma \in \mathcal{F}} : u_\sigma = 0 \text{ if } \sigma \subset \partial \Omega \}. \]

- $v \in X_{\mathcal{D},0} \mapsto \Pi_{\mathcal{D}} v$ linear.
  $\Pi_{\mathcal{D}} v : \Omega \to \mathbb{R}$ “reconstructed function”.

- $v \in X_{\mathcal{D},0} \mapsto \nabla_{\mathcal{D}} v$ linear.
  $\nabla_{\mathcal{D}} v : \Omega \to \mathbb{R}^d$ “reconstructed gradient”.
  $\| \nabla_{\mathcal{D}} v \|_{L^p(\Omega)}$ must be a norm on $X_{\mathcal{D},0}$. 
Step 2: the name of the game is substitution

**Gradient scheme**: in the weak formulation of the PDE, replace the space and operators by the discrete ones coming from $\mathcal{D}$. 
Step 2: the name of the game is substitution

**Gradient scheme:** in the weak formulation of the PDE, replace the space and operators by the discrete ones coming from $\mathcal{D}$.

**Linear diffusion:**

- Weak formulation:

$$\text{Find } \overline{u} \in H_0^1(\Omega) \text{ such that, } \forall \overline{v} \in H_0^1(\Omega),$$

$$\int_{\Omega} \nabla \overline{u} \cdot \nabla \overline{v} = \int_{\Omega} f \overline{v}.$$
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**Gradient scheme:** in the weak formulation of the PDE, replace the space and operators by the discrete ones coming from $D$.

**Linear diffusion:**

- Weak formulation:
  
  Find $\bar{u} \in H^1_0(\Omega)$ such that, $\forall \bar{v} \in H^1_0(\Omega)$,
  
  $$\int_\Omega \nabla \bar{u} \cdot \nabla \bar{v} = \int_\Omega f \bar{v}.$$ 

- Gradient scheme:
  
  Find $u \in X_{D,0}$ such that, $\forall v \in X_{D,0}$,
  
  $$\int_\Omega \nabla_D u \cdot \nabla_D v = \int_\Omega f \Pi_D v.$$
Step 2: the name of the game is substitution

**Gradient scheme:** in the weak formulation of the PDE, replace the space and operators by the discrete ones coming from $\mathcal{D}$.

**Semi-linear equation:** $-\Delta \bar{u} + \beta(\bar{u}) = f$

**Weak formulation:**

Find $\bar{u} \in H^1_0(\Omega)$ such that, $\forall \bar{v} \in H^1_0(\Omega)$,

$$
\int_{\Omega} \nabla \bar{u} \cdot \nabla \bar{v} + \int_{\Omega} \beta(\bar{u}) \bar{v} = \int_{\Omega} f \bar{v}.
$$
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Find $\overline{u} \in H^1_0(\Omega)$ such that, $\forall \overline{v} \in H^1_0(\Omega)$,

$$\int_{\Omega} \nabla \overline{u} \cdot \nabla \overline{v} + \int_{\Omega} \beta(\overline{u})\overline{v} = \int_{\Omega} f\overline{v}.$$ 

$\triangleright$ Gradient scheme:

Find $u \in X_{\mathcal{D},0}$ such that, $\forall v \in X_{\mathcal{D},0}$,

$$\int_{\Omega} \nabla_{\mathcal{D}} u \cdot \nabla_{\mathcal{D}} v + \int_{\Omega} \beta(\Pi_{\mathcal{D}} u)\Pi_{\mathcal{D}} v = \int_{\Omega} f\Pi_{\mathcal{D}} v.$$
Step 2: the name of the game is substitution

**Gradient scheme:** in the weak formulation of the PDE, replace the space and operators by the discrete ones coming from $\mathcal{D}$.

$p$-**Laplace equation:** $-\text{div}(|\nabla u|^{p-2}\nabla u) = f$

- Weak formulation:

  Find $\bar{u} \in W^{1,p}_0(\Omega)$ such that, $\forall \bar{v} \in W^{1,p}_0(\Omega)$,

  $$\int_{\Omega} |\nabla \bar{u}|^{p-2}\nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v}.$$
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**Gradient scheme**: in the weak formulation of the PDE, replace the space and operators by the discrete ones coming from $\mathcal{D}$.

**$\rho$-Laplace equation**: $-\text{div}(|\nabla \bar{u}|^{p-2} \nabla \bar{u}) = f$

- Weak formulation:

$$\text{Find } \bar{u} \in W_{0}^{1,p}(\Omega) \text{ such that, } \forall \bar{v} \in W_{0}^{1,p}(\Omega),$$

$$\int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v}.$$  

- Gradient scheme:

$$\text{Find } u \in X_{\mathcal{D},0} \text{ such that, } \forall v \in X_{\mathcal{D},0},$$

$$\int_{\Omega} |\nabla_{\mathcal{D}} u|^{p-2} \nabla_{\mathcal{D}} u \cdot \nabla_{\mathcal{D}} v = \int_{\Omega} f \Pi_{\mathcal{D}} v.$$
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3 measures of accuracy

Notation: \( \| g \|_{L^p} = \left( \int_\Omega |g|^p \right)^{1/p} \).

Measure of coercivity

\[
C_D = \max_{\nu_D \in X_D,0 \setminus \{0\}} \frac{\| \Pi_D \nu_D \|_{L^p}}{\| \nabla_D \nu_D \|_{L^p}}.
\]
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\[ C_D = \max_{\nu_D \in X_D, 0 \setminus \{0\}} \frac{\|\Pi_D \nu_D\|_{L^p}}{\|\nabla_D \nu_D\|_{L^p}}. \]

Measure of GD-consistency ("interpolation error" in FE)

\[ S_D(\varphi) = \min_{\nu_D \in X_D, 0} (\|\Pi_D \nu_D - \varphi\|_{L^p} + \|\nabla_D \nu_D - \nabla \varphi\|_{L^p}) . \]
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\]

Measure of GD-consistency ("interpolation error" in FE)

\[
S_D(\varphi) = \min_{\nu_D \in \mathcal{X}_D,0} (\|\Pi_D \nu_D - \varphi\|_{L^p} + \|\nabla_D \nu_D - \nabla \varphi\|_{L^p}).
\]

Measure of limit-confirmity ("consistency" in FE)

\[
W_D(\psi) = \max_{\nu_D \in \mathcal{X}_D,0 \setminus \{0\}} \frac{1}{\|\nabla_D \nu_D\|_{L^p}} \left| \int_{\Omega} \nabla_D \nu_D \cdot \psi + \Pi_D \nu_D \text{div} \psi \right|.
\]
Error estimates: Linear anisotropic and heterogeneous diffusion

With $\Lambda(x)$ symmetric positive definite matrix, bounded w.r.t. $x$,

- **Weak formulation:**

  Find $\overline{u} \in H^1_0(\Omega)$ such that, $\forall \overline{v} \in H^1_0(\Omega)$,

  $$
  \int_{\Omega} \Lambda \nabla \overline{u} \cdot \nabla \overline{v} = \int_{\Omega} f \overline{v}.
  $$

- **Gradient scheme:**

  Find $u \in X_{D,0}$ such that, $\forall v \in X_{D,0}$,

  $$
  \int_{\Omega} \Lambda \nabla_D u \cdot \nabla_D v = \int_{\Omega} f \Pi_D v.
  $$
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$$\int_{\Omega} \Lambda \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v}.$$

**Gradient scheme:**

Find $u \in X_{D,0}$ such that, $\forall v \in X_{D,0}$,

$$\int_{\Omega} \Lambda \nabla_D u \cdot \nabla_D v = \int_{\Omega} f \Pi_D v.$$

**Error estimate:**

$$\|\Pi_D u_D - \bar{u}\|_{L^2} + \|\nabla_D u_D - \nabla \bar{u}\|_{L^2} \leq C(1 + C_D) [S_D(\bar{u}) + W_D(\nabla \bar{u})].$$
Error estimates: $p$-Laplace

For $p \in (1, +\infty)$, with $W_0^{1,p}$ generalising $H^1_0$ to the power $p$ instead of 2 [no longer *Hilbert* space],

- **Weak formulation:**
  
  Find $\bar{u} \in W_0^{1,p}(\Omega)$ such that, $\forall \bar{v} \in W_0^{1,p}(\Omega)$,
  
  $$
  \int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v}.
  $$

- **Gradient scheme:**

  Find $u \in X_{D,0}$ such that, $\forall v \in X_{D,0}$,
  
  $$
  \int_{\Omega} |\nabla_D u|^{p-2} \nabla_D u \cdot \nabla_D v = \int_{\Omega} f \Pi_D v.
  $$

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Error estimates: \( p \)-Laplace

For \( p \in (1, +\infty) \), with \( W_0^{1,p} \) generalising \( H_0^1 \) to the power \( p \) instead of 2 [no longer Hilbert space],

\[ \text{Weak formulation:} \]

\[ \text{Find } \bar{u} \in W_0^{1,p}(\Omega) \text{ such that, } \forall \bar{v} \in W_0^{1,p}(\Omega), \]
\[ \int_{\Omega} |\nabla \bar{u}|^{p-2}\nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v}. \]

\[ \text{Gradient scheme:} \]

\[ \text{Find } u \in X_{D,0} \text{ such that, } \forall v \in X_{D,0}, \]
\[ \int_{\Omega} |\nabla_D u|^{p-2}\nabla_D u \cdot \nabla_D v = \int_{\Omega} f \Pi_D v. \]

\[ \text{Error estimate: If } p \in (1, 2], \]
\[ \|\Pi_D u_D - \bar{u}\|_{L^p} + \|\nabla_D u_D - \nabla \bar{u}\|_{L^p} \leq C(1+C_D) \left[ S_D(\bar{u}) + S_D(\bar{u})^{p-1} + W_D(|\nabla \bar{u}|^{p-2}\nabla \bar{u}) \right]. \]
Error estimates: $p$-Laplace

For $p \in (1, +\infty)$, with $W_0^{1,p}$ generalising $H_0^1$ to the power $p$ instead of 2 [no longer Hilbert space],

**Weak formulation:**

Find $\overline{u} \in W_0^{1,p}(\Omega)$ such that, $\forall v \in W_0^{1,p}(\Omega),$

$$\int_{\Omega} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla v = \int_{\Omega} f v.$$

**Gradient scheme:**

Find $u \in X_{D,0}$ such that, $\forall v \in X_{D,0},$

$$\int_{\Omega} |\nabla_D u|^{p-2} \nabla_D u \cdot \nabla_D v = \int_{\Omega} f \Pi_D v.$$

**Error estimate:** If $p \in [2, +\infty)$,

$$\|\Pi_D u_D - \overline{u}\|_{L^p} + \|\nabla_D u_D - \nabla \overline{u}\|_{L^p}$$

$$\leq C(1+C_D) \left[ S_D(\overline{u}) + \left[ S_D(\overline{u}) + W_D(|\nabla \overline{u}|^{p-2} \nabla \overline{u}) \right]^{\frac{1}{p-1}} \right].$$

J. Droniou (Monash University)
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Consider \((\mathcal{D}_m)_{m \in \mathbb{N}}\) sequence of GDs. From the previous estimates,

\[
\|\Pi_{\mathcal{D}_m} u_m - \bar{u}\|_{L^p} \to 0 \quad \text{and} \quad \|\nabla_{\mathcal{D}_m} u_m - \nabla \bar{u}\|_{L^p} \to 0
\]

provided that the following properties hold.

(P1) **Coercivity**: \((C_{\mathcal{D}_m})_{m \in \mathbb{N}}\) is bounded.

(P2) **GD-consistency**: for all \(\varphi \in W^{1,p}_0(\Omega)\), \(S_{\mathcal{D}_m}(\varphi) \to 0\) as \(m \to \infty\).

(P3) **Limit-conformity**: for all “proper” \(\psi : \Omega \to \mathbb{R}^d\), \(W_{\mathcal{D}_m}(\psi) \to 0\) as \(m \to \infty\).
Consider \((\mathcal{D}_m)_{m \in \mathbb{N}}\) sequence of GDs. From the previous estimates,

\[
\left\| \Pi_{\mathcal{D}_m} u_m - \bar{u} \right\|_{L^p} \to 0 \quad \text{and} \quad \left\| \nabla_{\mathcal{D}_m} u_m - \nabla \bar{u} \right\|_{L^p} \to 0
\]

provided that the following properties hold.

\begin{itemize}
  \item [(P1)] \textbf{Coercivity:} \((C_{\mathcal{D}_m})_{m \in \mathbb{N}}\) is bounded.
  
  \item [(P2)] \textbf{GD-consistency:} for all \(\varphi \in W_0^{1,p}(\Omega)\), \(S_{\mathcal{D}_m}(\varphi) \to 0\) as \(m \to \infty\).
  
  \item [(P3)] \textbf{Limit-conformity:} for all “proper” \(\psi : \Omega \to \mathbb{R}^d\), \(W_{\mathcal{D}_m}(\psi) \to 0\) as \(m \to \infty\).
\end{itemize}

\(\blacktriangleright\) Actually, \((P3) \implies (P1)\).
(P4) Compactness: $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is compact if for all $u_m \in X_{\mathcal{D}_m,0}$ such that $(\|\nabla_{\mathcal{D}_m} u_m\|_{L^p})_{m \in \mathbb{N}}$ is bounded, $(\Pi_{\mathcal{D}_m} u_m)_{m \in \mathbb{N}}$ is relatively compact in $L^p$.

(Discrete Rellich theorem).

▶ Useful for $-\text{div}(a(\bar{u})\nabla \bar{u}) = f$ for example.
(P5) Piecewise constant reconstruction: \( \mathcal{D} \) has a piecewise constant reconstruction if there exists a basis \((e_i)_{i \in I}\) of \(X_{\mathcal{D},0}\) and a partition \((\Omega_i)_{i \in I}\) of \(\Omega\) (some of them can be empty) such that, for all \( u = \sum_i u_i e_i \in X_{\mathcal{D},0}, \)

\[
\Pi_{\mathcal{D}} u = \sum_i u_i 1_{\Omega_i}.
\]

▶ Essential for \(-\Delta \bar{u} + \beta(\bar{u}) = f\) (comes from degenerate evolution problems).
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Conforming Galerkin approximations (e.g. \( P_1 \))

Gradient discretisation:

- \( X_{D,0} \) finite dimensional subspace of \( W_0^{1,p}(\Omega) \),
- \( \Pi_D = \text{Id} : X_{D,0} \rightarrow W_0^{1,p}(\Omega) \),
- \( \nabla_D = \nabla : X_{D,0} \rightarrow W_0^{1,p}(\Omega) \).

Properties:

(P1) Coercivity: \( C_D \leq \text{Poincaré constant in } W_0^{1,p}(\Omega) \),

(P2) GD-consistency: if \( \bigcup_{m \in \mathbb{N}} X_{D,m},0 \) "ultimately dense" in \( W_0^{1,p}(\Omega) \),

(P3) Limit-conformity:

(P4) Compactness: Rellich theorem in \( W_0^{1,p}(\Omega) \),

(P5) Piecewise-constant: No \( \rightarrow \) modify into mass-lumped version (no longer conforming).
Conforming Galerkin approximations (e.g. $P_1$)

Gradient discretisation:

- $X_{D,0}$ finite dimensional subspace of $W_0^{1,p}(\Omega)$,
- $\Pi_D = \text{Id} : X_{D,0} \rightarrow W_0^{1,p}(\Omega)$,
- $\nabla_D = \nabla : X_{D,0} \rightarrow W_0^{1,p}(\Omega)$.

Properties:

- (P1) Coercivity: $C_D \leq$ Poincaré constant in $W^{1,p}(\Omega)$,
- (P2) GD-consistency: if $\bigcup_{m \in \mathbb{N}} X_{D_m,0}$ “ultimately dense” in $W^{1,p}(\Omega)$,
- (P3) Limit-conformity: $W_D \equiv 0$,
- (P4) Compactness: Rellich theorem in $W_0^{1,p}(\Omega)$,
- (P5) Piecewise-constant: No $\Rightarrow$ modify into mass-lumped version (no longer conforming).
Conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh.

**Gradient discretisation:**

- $X_{D,0} = \{ u = (u_v)_v \text{ vertex} : u_v = 0 \text{ if } v \in \partial\Omega \}$,
- $\Pi_D : X_{D,0} \to W_0^{1,p}(\Omega)$ such that for all $K \in \mathcal{M}$, $(\Pi_D u)|_K = \text{affine map with value } u_v \text{ at any vertex } v \text{ of } K$.
- $\nabla_D : X_{D,0} \to W_0^{1,p}(\Omega)$ such that $(\nabla_D u)|_K = \nabla(\Pi_D u)|_K$ (broken gradient).
Mass-lumped conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh.

**Gradient discretisation:**

- $X_{D,0}, \nabla_D$ as for $\mathbb{P}_1$ finite elements.
- $\Pi_D u = u_\nu$ on a “dual” cell around $\nu$.

$$\Omega \quad \Pi_D u$$

$$\begin{align*}
(\nabla_D u)_K &\neq \nabla (\Pi_D u)_K.
\end{align*}$$
Non-conforming $P_1$ finite elements

On a triangular/tetrahedral mesh.

**Gradient discretisation:**

- $X_{D,0} = \{u = (u_\sigma)_{\sigma \in \mathcal{F}} : u_\sigma = 0 \text{ if } \sigma \in \mathcal{F}_{\text{ext}}\}$,
- $\Pi_D : X_{D,0} \to W_0^{1,p}(\Omega)$ such that for all $K \in \mathcal{M}$, $(\Pi_D u)|_K = \text{affine map with value } u_\sigma \text{ at } \bar{x}_\sigma \text{ for all } \sigma \in \mathcal{F}_K$.
- $\nabla_D : X_{D,0} \to W_0^{1,p}(\Omega)$ such that $(\nabla_D u)|_K = \nabla(\Pi_D u)|_K$ (broken gradient).
Mass-lumped non-conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh.

**Gradient discretisation:**

- $X_{\mathcal{D},0}$ and $\nabla_{\mathcal{D}}$ as for non-conforming finite elements,
- $\Pi_{\mathcal{D}} u = u_\sigma$ on a “dual” cell around $\sigma$.

\[ (\nabla_{\mathcal{D}} u)_{|K} \neq \nabla (\Pi_{\mathcal{D}} u)_{|K}. \]
\( \mathcal{I} = \text{Cartesian mesh (also possible with triangular/tetrahedral).} \)

**Gradient discretisation:**

- \( u \in X_{D,0} \) if \( u = ((u_K)_K, (u_{\sigma, v})_{\sigma, v}) \) with \( K \) cells and \( (\sigma, v) \) pairs edge-vertex s.t. \( v \in \sigma \), and \( u_{\sigma, s} = 0 \) if \( \sigma \in F_{\text{ext}} \).

- \( \Pi_D u = u_K \) in \( K \),

- \( \nabla_D u = \frac{u_{\sigma, v} - u_K}{d(x_K, \sigma)} \mathbf{n}_{K,\sigma} + \frac{u_{\tau, v} - u_K}{d(x_K, \tau)} \mathbf{n}_{K,\tau} \) in the cube defined by \( K \) and \( v \).
Other methods known to be GDMs:

- Any non-conforming FE method, including non-conforming $P_k$.
- Mixed finite elements, including $\mathbb{RT}_k$ (leads to $H_{\text{div}}$-conforming gradient discretisations: $\nabla_D u \in H_{\text{div}}$).
- SIPG discontinuous Galerkin method.
- Hybrid Mimetic Mixed methods, including mixed-hybrid Mimetic Finite Differences.
- Nodal Mimetic Finite Difference methods.
- Hybrid high-order methods, non-conforming Virtual Element Methods, non-conforming Mimetic Finite Difference.
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Tools for mesh-based GDM

Polytopal toolbox
▶ Provides generic theorems that make the proof of (P1) coercivity, (P3) limit-conformity, and (P4) compactness very easy.

Local linearly exact gradient discretisation:
▶ Rigorous notion for “$\nabla_D$ exactly reconstructs linear functions” *(nearly all numerical methods try to satisfy this property).*
▶ Provides easy proof of (P2) consistency for all methods.
Proof of (P1)–(P4) for non-conforming $\mathbb{P}_1$ gradient discretisations. We drop the index $m$ from time to time for sake of legibility, and all constants below do not depend on $m$ or the considered cells/edges. Let us define a control of $D$ by $T$ in the sense of Definition 2.29, where $T$ is the simplicial mesh associated to $D$, with $x_K = \bar{x}_K = \frac{1}{d+1} \sum_{\sigma \in E_K} \bar{x}_\sigma$ the centres of gravity of the cells $K$. We define the linear (injective) mappings $\Phi : X_{D_m,0} \to X_{T_m,0}$ by $\Phi(u)_K = \frac{1}{d+1} \sum_{\sigma \in E_K} u_\sigma = \Pi_D u(x_K)$ and $\Phi(u)_\sigma = u_\sigma = \Pi_D u(\bar{x}_\sigma)$.

Since $\Phi(u)_K = \Pi_D u(x_K)$ and $G_K u = \nabla(\Pi_D u)$ in $K$, we get

$$\Phi(u)_\sigma - \Phi(u)_K = G_K u \cdot (\bar{x}_\sigma - x_K). \tag{3.2}$$

Therefore, since $\frac{|\bar{x}_\sigma - x_K|}{d_{K,\sigma}} \leq \frac{h_K}{d_{K,\sigma}} \leq \theta_T$,

$$\sum_{\sigma \in E_K} |\sigma|d_{K,\sigma} \left| \frac{\Phi(u)_\sigma - \Phi(u)_K}{d_{K,\sigma}} \right|^p \leq \theta_T^p d|K| |G_K u|^p.$$ 

This implies (2.34). We now observe that the affine function $\alpha_\sigma$ reaches its extremal values at the vertices of $K$. It is easy to see that $\alpha_\sigma(\nu_\sigma) = 1 - d$, where $\nu_\sigma$ is the vertex opposite to the face $\sigma$, and that $\alpha_\sigma(\nu_{\sigma'}) = 1$ for all $\sigma' \neq \sigma$. Therefore, for $x \in K$,

$$|\Pi_D u(x) - \Phi(u)_K| = \left| \sum_{\sigma \in E_K} (\Phi(u)_\sigma - \Phi(u)_K) \alpha_\sigma(x) \right| \leq (d + 1) \max(1, d - 1) \max_{\sigma \in E_K} |G_K u \cdot (\bar{x}_\sigma - x_K)|.$$ 

This inequality implies $\omega^{\Pi}(D, T, \Phi) \leq (d + 1) \max(1, d - 1) h_M$ and therefore (2.35) holds. Finally, recalling that $\Pi_D u$ is affine in each simplex $K$ and that $\nabla_T$ is exact on interpolants of affine functions (cf. Lemma 2.28), we see that $\nabla_D u = \nabla_T \Phi(u)$ in $\Omega$. Hence $\omega^{\nabla}(D, T, \Phi) = 0$ and (2.36) holds. Proposition 2.31 therefore shows that $(D_m)_{m \in \mathbb{N}}$ is coercive in the sense of Definition 2.2, limit-conforming in the sense of Definition 2.4, and compact in the sense of Definition 2.5.

Since non-conforming $\mathbb{P}_1$ gradient discretisations are LLE gradient discretisations, the consistency of $(D_m)_{m \in \mathbb{N}}$ follows from Proposition 2.14 by noticing that $\text{reg}_{\text{LLE}}(D_m)$ is controlled by $\theta_{T_m}$. \hfill $\square$
Proof of the property (P) for MPFA-O gradient discretisations. We drop the indices \( m \) for sake of legibility. We consider the polytopal mesh \( \mathcal{T} = (\mathcal{M}, \mathcal{E}', \mathcal{P}, \mathcal{V}') \) where the sets \((\mathcal{M}, \mathcal{P})\) are those of the original polytopal mesh, \( \mathcal{E}' = \{ \sigma_v \mid \sigma \in \mathcal{E}, \, v \in \mathcal{V} \} \), and \( \mathcal{V}' \) is the set of all vertices of the elements of \( \mathcal{E}' \). We define a control of \( \mathcal{D} \) by \( \mathcal{T} \) in the sense of Definition 2.29 as the isomorphism \( \Phi : X_{\mathcal{D}, 0} \longrightarrow X_{\mathcal{T}, 0} \) given by \( \Phi(u)_K = u_K \) and \( \Phi(u)_{\sigma_v} = u_{(\sigma, v)} \). We observe that

\[
\int_K |\nabla_D u(x)|^p \, dx \geq C_3 \sum_{\sigma \in \mathcal{E}_K} \sum_{v \in \mathcal{V}_\sigma} |\sigma_v| d_{K, \sigma} \left| \frac{u_{(\sigma, v)} - u_K}{d_{K, \sigma}} \right|^p,
\]

with \( C_3 = 1 \) for parallelepipedic meshes, and \( C_3 > 0 \) depends on an upper bound of the regularity of the mesh for simplicial meshes. Therefore \( \|\nabla_D u\|_{L^p(\Omega)^d}^p \geq C_3 \|\Phi(u)\|_{T, 0, p}^p \) and (2.34) is proved. Since \( \Pi_D u = \Pi_T \Phi(u) \), we get \( \omega^\Pi(\mathcal{D}, \mathcal{T}, \Phi) = 0 \), which proves (2.35). Finally, we have

\[
\int_K \nabla_D u(x) \, dx = \sum_{\sigma \in \mathcal{E}_K} \sum_{v \in \mathcal{V}_\sigma} |\sigma_v| (u_{\sigma,v} - u_K) n_{K, \sigma} = \sum_{\sigma' \in \mathcal{E}'_K} |\sigma'| (\Phi(u)_{\sigma'} - \Phi(u)_K) n_{K, \sigma'} = |K| \nabla_T \Phi(u)|_K.
\]

This shows that \( \omega^\nabla(\mathcal{D}, \mathcal{T}, \Phi) = 0 \), which establishes (2.36). Proposition 2.31 therefore shows that \( (\mathcal{D}_m)_{m \in \mathbb{N}} \) is coercive in the sense of Definition 2.2, limit-conforming in the sense of Definition 2.4, and compact in the sense of Definition 2.5.

It is proved in \([40, 41]\) that the definitions of the approximation points \( S \) give the LLE property in both the Cartesian and simplicial cases. Hence, the consistency of \( (\mathcal{D}_m)_{m \in \mathbb{N}} \) follows from Proposition 2.14. \( \square \)
Proof of the property \((P)\) for HMM gradient discretisations. Let \((\mathcal{D}_m)_{m \in \mathbb{N}}\) be HMM gradient discretisations built on polytopal meshes \((\mathcal{T}_m)_{m \in \mathbb{N}}\), and let us define a control of \(\mathcal{D}_m\) by \(\mathcal{T}_m\) in the sense of Definition 2.29. We drop the index \(m\) from time to time. Since \(X_{\mathcal{D},0} = X_{\mathcal{T},0}\), we can take \(\Phi = \text{Id}\). Estimate (2.34) is given by (3.14). Relation (2.35) follows immediately since \(\omega^\Pi(\mathcal{D}, \mathcal{T}, \Phi) = 0\), owing to \(\Pi_{\mathcal{D}} u = \Pi_{\mathcal{T}} u = \Pi_{\mathcal{T}} \Phi(u)\). Recalling that \(|D_{K,\sigma}| = \frac{\sigma|d_{K,\sigma}|}{d}\) we have

\[
\int_K \nabla_D u(x) dx = |K| \nabla_K u + \frac{1}{\sqrt{d}} \sum_{\sigma \in \mathcal{E}_K} |\sigma| [\mathcal{L}_K R_K(Q_K(u))]_{\sigma} n_{K,\sigma}. \tag{3.15}
\]

The definition of \(R_K\) and the property \(\sum_{\sigma \in \mathcal{E}_K} |\sigma| n_{K,\sigma}(\bar{x}_\sigma - x_K)^T = |K| \text{Id}\) (a consequence of Stokes’ formula) show that for any \(\eta \in \text{Im}(R_K)\) we have \(\sum_{\sigma \in \mathcal{E}_K} |\sigma| \eta_{\sigma} n_{K,\sigma} = 0\). Hence, since \(\text{Im}(\mathcal{L}_K) = \text{Im}(R_K)\), (3.15) gives

\[
\int_K \nabla_D u(x) dx = |K| \nabla_K u = |K| \nabla_T \Phi(u)|_K,
\]

which shows that \(\omega^\nabla(\mathcal{D}, \mathcal{T}, \Phi) = 0\), and thus that (2.36) holds. The coercivity, limit-conformity and compactness of \((\mathcal{D}_m)_{m \in \mathbb{N}}\) therefore follow from Proposition 2.31. Since HMM gradient discretisations are LLE gradient discretisations, the consistency of \((\mathcal{D}_m)_{m \in \mathbb{N}}\) readily follows from Proposition 2.14, after noticing that the regularity assumption on \((\mathcal{D}_m)_{m \in \mathbb{N}}\) gives a bound on \((\text{reg}_{\text{LLE}}(\mathcal{D}_m))_{m \in \mathbb{N}}\). \(\square\)
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Convergence analysis by compactness

I. A priori estimates: prove that approximation solutions $u_h$ remain bounded in certain norms.

II. Compactness result: from above estimates, prove convergence up to subsequence of $u_h$ to some $\bar{u}$.

III. Proof that $\bar{u}$ is a solution to the model: “pass to the limit” in the scheme (plug interpolant of smooth test functions) to prove that $\bar{u}$ is a weak solution of the continuous model.
Convergence analysis by compactness

I. A priori estimates: prove that approximation solutions $u_h$ remain bounded in certain norms.

GDM: energy estimates ($\times \overline{u}$, integrate) automatic if the continuous model satisfies them.

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GDM: provides generic compactness theorems for that, including for time-dependent problems.

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III. **Proof that $\bar{u}$ is a solution to the model**: “pass to the limit” in the scheme (plug interpolant of smooth test functions) to prove that $\bar{u}$ is a weak solution of the continuous model.

- GDM: provides generic interpolation result for test functions.
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TPFA method for $-\Delta \bar{u} = f$

- Used since the 50’ by petroleum engineers.
Used since the 50’ by petroleum engineers.

Approximate $\bar{u}$ by $u_h$ piecewise constant on an “admissible” mesh $\mathcal{T}$.

Classical example: triangular mesh with $x_K =$ circumcenter of $K$.

Expected and proved error estimate: $\|u_h - \overline{u}\|_{L^2} = O(h)$, if $\bar{u}$ smooth enough.
TPFA method for $-\Delta \overline{u} = f$

- Used since the 50's by petroleum engineers.

- Approximate $\overline{u}$ by $u_h$ piecewise constant on an "admissible" mesh $\mathcal{T}$.

  Classical example: triangular mesh with $x_K =$ circumcenter of $K$.

- Expected and proved error estimate: $\|u_h - \overline{u}\|_{L^2} = O(h)$, if $\overline{u}$ smooth enough.

- Experimentally observed: if $P_h \overline{u}$ piecewise constant function equal to $\overline{u}(x_K)$ on $K \in \mathcal{M}$,

$$\|u_h - P_h \overline{u}\|_{L^2} = O(h^2).$$

Under $H^2$ regularity assumption on the PDE, on sequences of triangular meshes as above,

\[ \| u_h - P_h u \|_{L^2} = O(h^2) \].

\[ \text{▶ Very indirect proof (exploits relation HMM–TPFA, skewed projection in } L^2 \text{, and local compensation properties of the meshes...).} \]

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Under $H^2$ regularity assumption on the PDE, on sequences of triangular meshes as above,

$$\|u_h - P_h u\|_{L^2} = \mathcal{O}(h^2).$$

Very indirect proof (exploits relation HMM–TPFA, skewed projection in $L^2$, and local compensation properties of the meshes...).
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Cost functional: $J : \Omega \times \mathbb{R} \times \mathbb{R}^d \to [0, +\infty)$ convex w.r.t. third variable and

$$J(x, s, \xi) \geq \alpha|\xi|^2 - \beta(x)$$

Minimisation problem:

Find $\bar{u} \in H^1_0(\Omega)$ that realises

$$\min_{v \in H^1_0(\Omega)} \int_{\Omega} J(x, v(x), \nabla v(x)) \, dx.$$
Model: continuous optimisation problem

Cost functional: \( J : \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow [0, +\infty) \) convex w.r.t. third variable and
\[
J(x, s, \xi) \geq \alpha|\xi|^2 - \beta(x)
\]

Minimisation problem:

Find \( \bar{u} \in H_0^1(\Omega) \) that realises \[
\min_{v \in H_0^1(\Omega)} \int_{\Omega} J(x, v(x), \nabla v(x)) \, dx.
\]

GDM approximation: the name of the game is substitution...

Find \( u \in X_{D,0} \) that realises \[
\min_{v \in X_{D,0}} \int_{\Omega} J(x, \Pi_D v(x), \nabla_D v(x)) \, dx.
\]

➤ Covers a wide range of numerical schemes at once...
Convergence by compactness

Take \((D_m)_{m \in \mathbb{N}}\) that satisfies (P1)–(P4).

**A priori estimates:** using \(0 \in X_{D,0},\)

\[
\alpha \| \nabla D_m u_m \|^2_{L^2} - \beta \leq \int_{\Omega} J(x,0,0) \, dx
\]

so \((\| \nabla D_m u_m \|_{L^2})_{m \in \mathbb{N}}\) bounded.
Convergence by compactness

Take \((\mathcal{D}_m)_{m \in \mathbb{N}}\) that satisfies (P1)–(P4).

**A priori estimates:** \((\|\nabla_{\mathcal{D}_m} u_m\|_{L^2})_{m \in \mathbb{N}}\) bounded.

**Compactness results:** generic GDM results show that, up to a subsequence,
- \(\Pi_{\mathcal{D}_m} u_m \rightarrow \overline{u}\) strongly in \(L^2\), with \(\overline{u} \in H^1_0(\Omega)\),
- \(\nabla_{\mathcal{D}_m} u_m \rightarrow \nabla \overline{u}\) weakly in \(L^2\).
Take \((\mathcal{D}_m)_{m\in\mathbb{N}}\) that satisfies (P1)–(P4).

**A priori estimates:** \((\|\nabla_{\mathcal{D}_m} u_m\|_{L^2})_{m\in\mathbb{N}}\) bounded.

**Compactness results:** up to a subsequence,

- \(\Pi_{\mathcal{D}_m} u_m \rightarrow \bar{u}\) strongly in \(L^2\), with \(\bar{u} \in H^1_0(\Omega)\),
- \(\nabla_{\mathcal{D}_m} u_m \rightarrow \nabla \bar{u}\) weakly in \(L^2\).

**Limit problem:** take \(\varphi \in H^1_0(\Omega)\). Generic GDM result give an interpolant \(I_{\mathcal{D}_m}\varphi \in X_{\mathcal{D}_m,0}\) such that

\[
\Pi_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi \rightarrow \varphi \quad \text{and} \quad \nabla_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi \rightarrow \nabla \varphi \quad \text{strongly in} \quad L^2.
\]
Convergence by compactness

Take \((D_m)_{m \in \mathbb{N}}\) that satisfies (P1)–(P4).

**A priori estimates:** \((\|\nabla_{D_m} u_m\|_{L^2})_{m \in \mathbb{N}}\) bounded.

**Compactness results:** up to a subsequence,
- \(\Pi_{D_m} u_m \to \bar{u}\) strongly in \(L^2\), with \(\bar{u} \in H^1_0(\Omega)\),
- \(\nabla_{D_m} u_m \to \nabla \bar{u}\) weakly in \(L^2\).

**Limit problem:** take \(\varphi \in H^1_0(\Omega)\).

\[\Pi_{D_m} l_{D_m} \varphi \to \varphi\text{ and }\nabla_{D_m} l_{D_m} \varphi \to \nabla \varphi\text{ strongly in }L^2.\]

By definition of \(u_m\),
\[
\int_{\Omega} J(x, \Pi_{D_m} u_m, \nabla_{D_m} u_m) \, dx \leq \int_{\Omega} J(x, \Pi_{D_m} l_{D_m} \varphi, \nabla_{D_m} l_{D_m} \varphi) \, dx.
\]
Take \((\mathcal{D}_m)_{m \in \mathbb{N}}\) that satisfies (P1)–(P4).

**A priori estimates:** \((\|\nabla_{\mathcal{D}_m} u_m\|_{L^2})_{m \in \mathbb{N}}\) bounded.

**Compactness results:** up to a subsequence,
- \(\Pi_{\mathcal{D}_m} u_m \to \bar{u}\) strongly in \(L^2\), with \(\bar{u} \in H^1_0(\Omega)\),
- \(\nabla_{\mathcal{D}_m} u_m \to \nabla \bar{u}\) weakly in \(L^2\).

**Limit problem:** take \(\varphi \in H^1_0(\Omega)\).

\[
\Pi_{\mathcal{D}_m} I_{\mathcal{D}_m} \varphi \to \varphi \quad \text{and} \quad \nabla_{\mathcal{D}_m} I_{\mathcal{D}_m} \varphi \to \nabla \varphi \quad \text{strongly in} \quad L^2.
\]

By definition of \(u_m\),

\[
\int_{\Omega} J(x, \Pi_{\mathcal{D}_m} u_m, \nabla_{\mathcal{D}_m} u_m) \, dx \leq \int_{\Omega} J(x, \Pi_{\mathcal{D}_m} I_{\mathcal{D}_m} \varphi, \nabla_{\mathcal{D}_m} I_{\mathcal{D}_m} \varphi) \, dx.
\]

- RHS: strong convergences of \(\Pi_{\mathcal{D}_m} I_{\mathcal{D}_m}\) and \(\nabla_{\mathcal{D}_m} I_{\mathcal{D}_m} \varphi\).
- LHS: strong convergence of \(\Pi_{\mathcal{D}_m} u_m\), weak convergence of \(\nabla_{\mathcal{D}_m} u_m\) and convexity of \(J\) w.r.t. third variable.
Convergence by compactness

Take \((D_m)_{m \in \mathbb{N}}\) that satisfies (P1)–(P4).

**A priori estimates:** \((\|\nabla D_m u_m\|_{L^2})_{m \in \mathbb{N}}\) bounded.

**Compactness results:** up to a subsequence,
- \(\Pi D_m u_m \to \bar{u}\) strongly in \(L^2\), with \(\bar{u} \in H_0^1(\Omega)\),
- \(\nabla D_m u_m \to \nabla \bar{u}\) weakly in \(L^2\).

**Limit problem:** take \(\varphi \in H_0^1(\Omega)\).

\[\Pi D_m I_{D_m} \varphi \to \varphi\text{ and }\nabla D_m I_{D_m} \varphi \to \nabla \varphi\text{ strongly in }L^2.\]

\[
\int_{\Omega} J(x, \bar{u}, \nabla \bar{u}) \, dx \leq \int_{\Omega} J(x, \varphi, \nabla \varphi) \, dx.
\]
Conclusion: tools for convergence analysis in real-world situations

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Bibliography

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Thanks.