A unified convergence analysis framework for numerical approximations of diffusion equations

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Joint work with many collaborators...

Australian Government
Australian Research Council

How are you with combinatorics?

If we are to analyse the convergence of each numerical method for each model:

Methods	Models
FE	Heat equation
MFE	Miscible flow
MPFA	Multi-phase flow
MFV	Stefan problem
DDFV	Image processing
MFD	Richards equation
dG	Navier-Stokes
...	...

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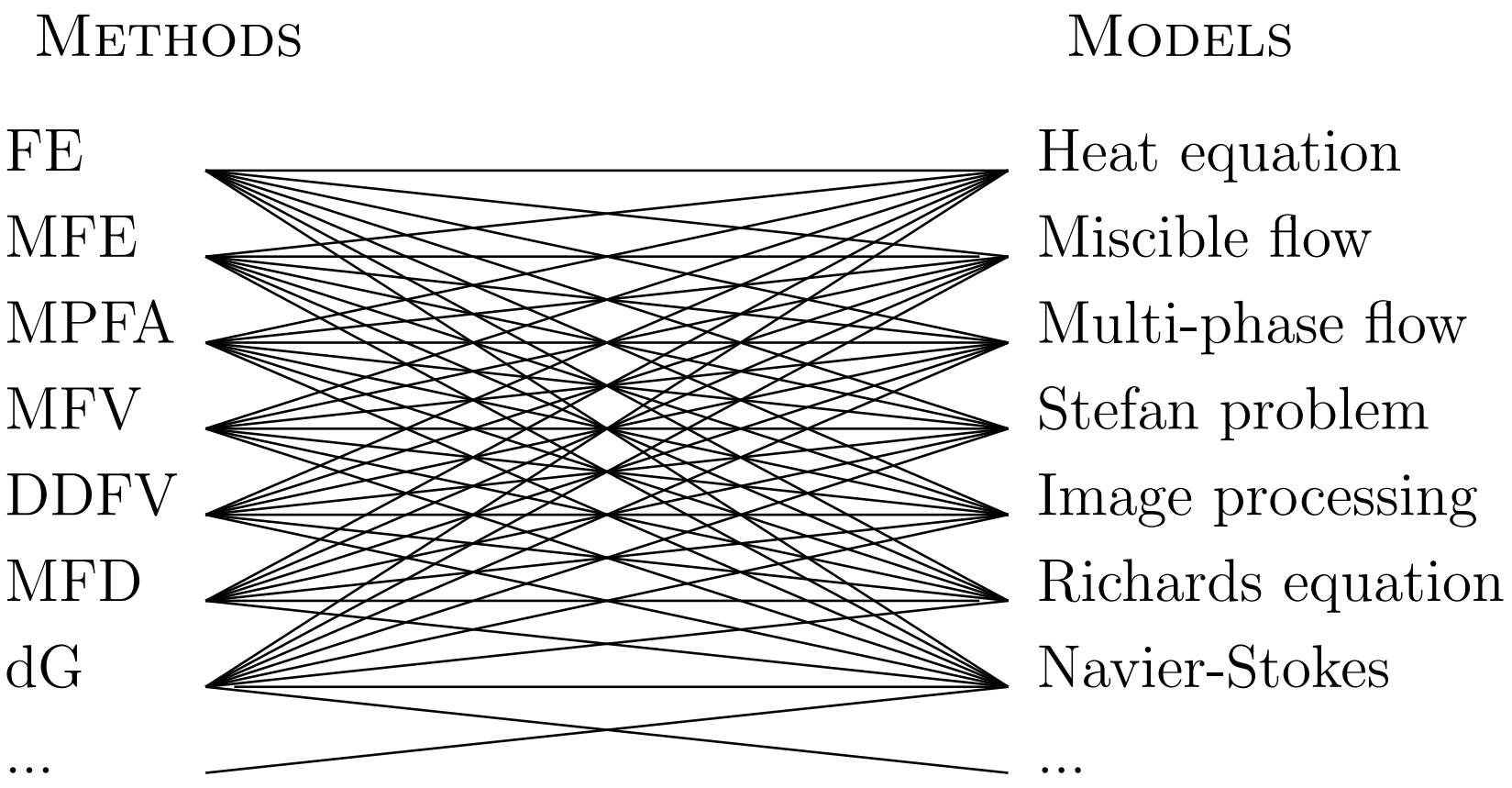
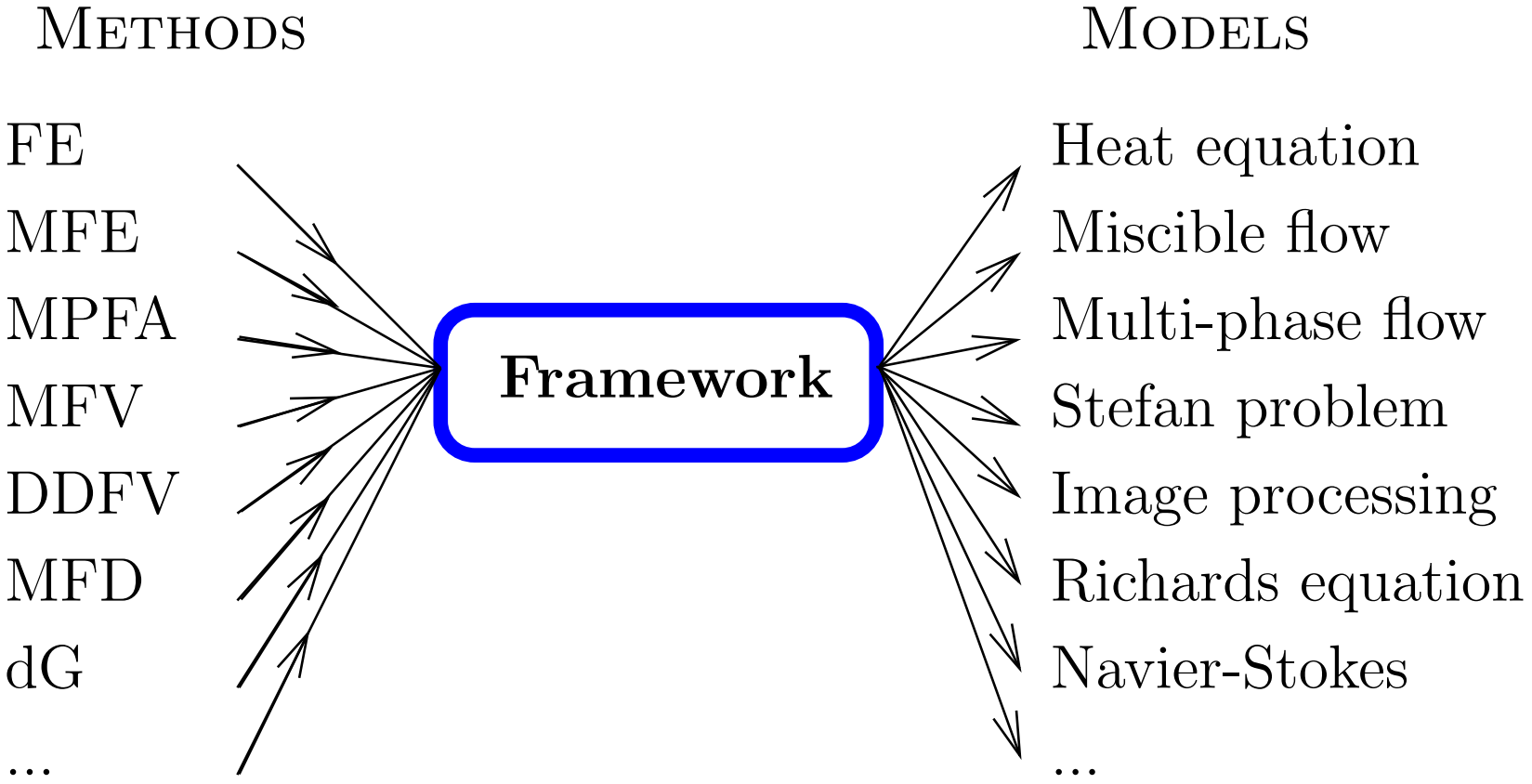


Each line = one analysis to perform... add to that all the benchmarking!

How about going through some sort of generic framework?



Objective: the framework identifies a few key properties that all methods satisfy, and that are sufficient for all convergence analyses.

Plan

Laplace equation

$$\begin{cases} -\Delta \overline{u} = f & \text{in } \Omega \\ \overline{u} = 0 & \text{on } \partial\Omega \end{cases}$$

- Ω open bounded in $\mathbb{R}^d$,
- $f : \Omega \to \mathbb{R}$.

▶ Prototypal of diffusion models, e.g. heat equation, flows in porous media, etc.

Weak formulation of the Laplace equation

$$\begin{cases} -\Delta \overline{u} = f & \text{in } \Omega \\ \overline{u} = 0 & \text{on } \partial\Omega \end{cases}$$

Divergence formula (integration-by-parts): for $F : \Omega \to \mathbb{R}^d$ and $v : \Omega \to \mathbb{R}$, if $\mathbf{n}$ is the outer normal to $\partial\Omega$,

$$\int_\Omega v \mathrm{div}(F) = \int_{\partial\Omega} vF \cdot \mathbf{n} - \int_\Omega F \cdot \nabla v$$

where $\nabla v = (\partial_1 v, \ldots, \partial_d v)$.

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For Laplace equation: since $\Delta \overline{u} = \mathrm{div}(\nabla \overline{u})$, multiply equation by $\overline{v}$ such that $\overline{v} = 0$ on $\partial\Omega$ and integrate by parts:

$$\int_\Omega \nabla \overline{u} \cdot \nabla \overline{v} = \int_\Omega f \overline{v}.$$

Weak formulation of the Laplace equation

$$\overline{u} = 0 \text{ on } \partial\Omega \text{ such that, for all } \overline{v} = 0 \text{ on } \partial\Omega,$$
$$\int_\Omega \nabla\overline{u} \cdot \nabla\overline{v} = \int_\Omega f\overline{v}.$$

Riesz representation theorem: If $\langle \cdot, \cdot \rangle$ is an inner product in $\mathbb{R}^N$, for any $\ell : \mathbb{R}^N \to \mathbb{R}$ linear, there is a unique solution to

$$\text{Find } X \in \mathbb{R}^N \text{ s.t., for all } Y \in \mathbb{R}^N, \quad \langle X, Y \rangle = \ell(Y).$$

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"Application" to Laplace: define

$$\langle \overline{u}, \overline{v} \rangle = \int_\Omega \nabla\overline{u} \cdot \nabla\overline{v}\,, \qquad \ell(\overline{v}) = \int_\Omega f\overline{v}.$$

- $\overline{u}, \overline{v}$ are not vectors in $\mathbb{R}^N$, but functions (infinite dimensional space!).

Weak formulation

Hilbert space: Riesz representation theorem is also valid in *Hilbert* spaces.

$$H_0^1(\Omega) = \Big\{ \overline{v} : \Omega \to \mathbb{R} \, : \int_\Omega |\overline{v}|^2 < +\infty \, , \; \int_\Omega |\nabla \overline{v}|^2 < +\infty \, , \\ \overline{v} = 0 \text{ on } \partial\Omega \Big\}$$

with inner product $\langle \overline{u}, \overline{v} \rangle = \displaystyle\int_\Omega \nabla \overline{u} \cdot \nabla \overline{v}$.

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- Existence and uniqueness.

Plan

Galerkin approximation

$$\text{Find } \overline{u} \in H_0^1(\Omega) \text{ s.t., for all } \overline{v} \in H_0^1(\Omega), \quad \int_\Omega \nabla \overline{u} \cdot \nabla \overline{v} = \int_\Omega f \overline{v}.$$

Computational issue: $H_0^1(\Omega)$ is an infinite-dimensional space.

$\rightsquigarrow$ Cannot be understood/manipulated by computer.

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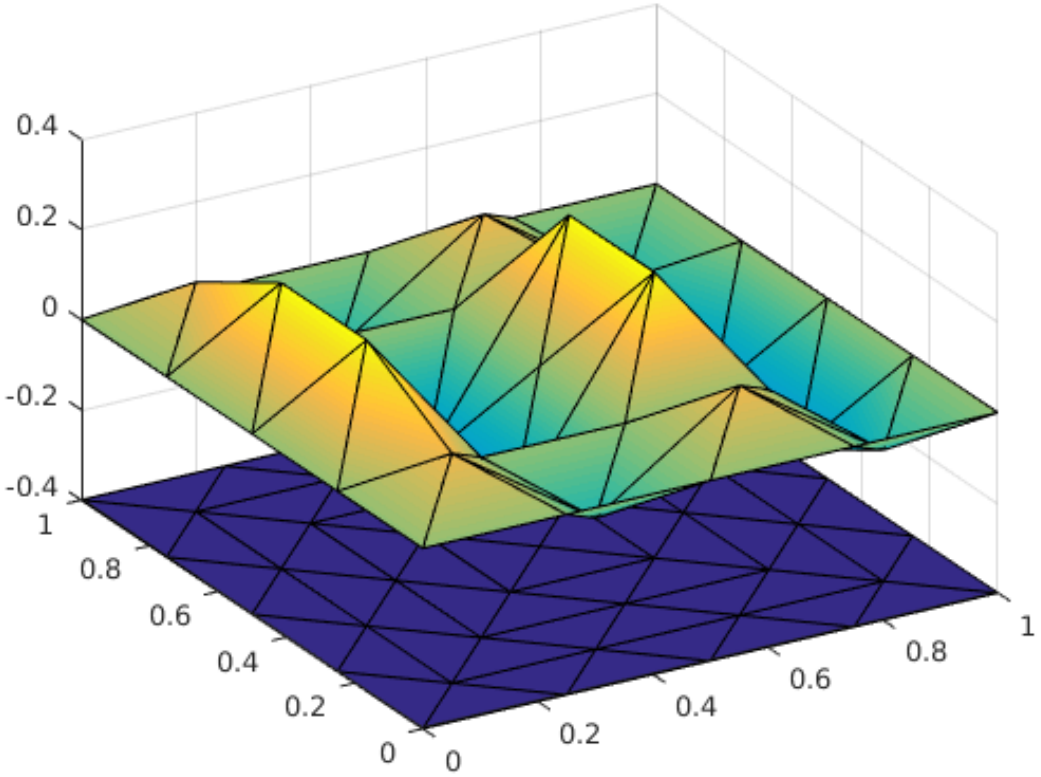
Easy solution: take $E \subset H_0^1(\Omega)$ finite-dimensional subspace and write

$$\text{Find } u_E \in E \text{ s.t., for all } v_E \in E, \quad \int_\Omega \nabla u_E \cdot \nabla v_E = \int_\Omega f v_E.$$

Example: $\mathbb{P}_1$ finite elements

Space E: cut Ω into triangles and consider continuous functions that are affine in each triangle (and zero on the boundary).



- Described by values at the vertices (finite number of such values).

Non-conforming $\mathbb{P}_1$

Non-conforming approximation: $E \not\subset H_0^1(\Omega)$.

$\rightsquigarrow$ Define ∇v_E if $v_E \in E$?

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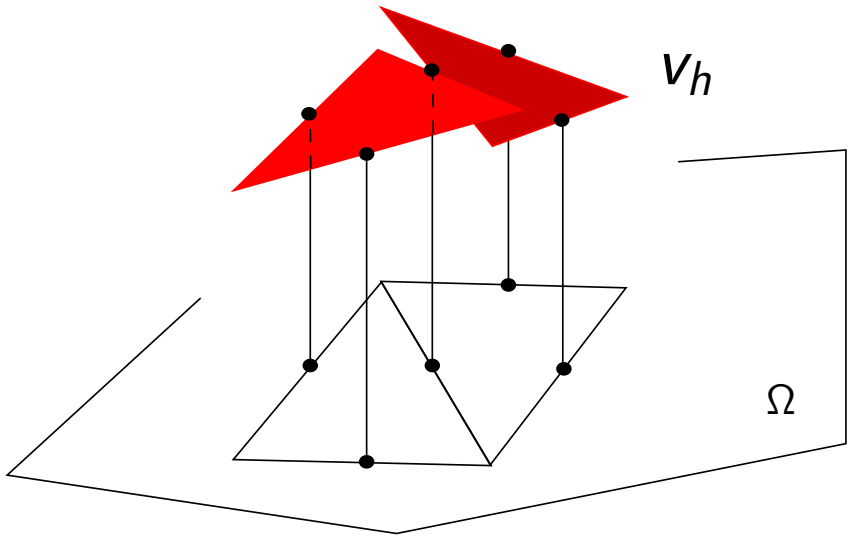
Example: non-conforming $\mathbb{P}_1$ on a triangular mesh:

$$E = E_h = \{v_h : \Omega \to \mathbb{R} \ : \ v_h \text{ piecewise linear, continuous at edge midpoints, zero at boundary edge midpoints}\}$$



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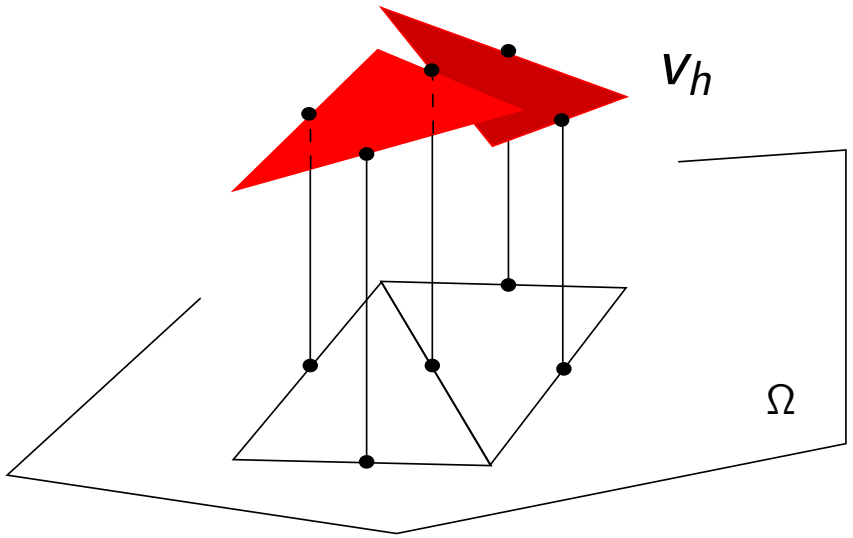
$\rightsquigarrow$ Define ∇v_E if $v_E \in E$?

Example: non-conforming $\mathbb{P}_1$ on a triangular mesh:

- ∇u_h replaced with broken gradient $\nabla_h u_h$, computed cell-wise.

Find $u_h \in E_h$ s.t., for all $v_h \in E_h$, $\displaystyle\int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h$.



Mass-lumping

- Non-linear diffusion-reaction (*e.g., from Euler time-stepping on Richards' equation*): with $\beta(s)s \geq 0$,

$$-\Delta \overline{u} + \beta(\overline{u}) = f.$$

- Weak formulation:

$$\overline{u} \in H_0^1(\Omega)\,,\ \forall v \in H_0^1(\Omega)\,,\ \int_\Omega \nabla \overline{u} \cdot \nabla \overline{v} + \int_\Omega \beta(\overline{u})\overline{v} = \int_\Omega f\overline{v}.$$

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Numerical approximation: e.g., non-conforming $\mathbb{P}_1$ finite elements: find $u_h \in E_h$ s.t., $\forall v_h \in E_h$,

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h + \begin{cases} \displaystyle\int_\Omega \beta(u_h) v_h \\ \displaystyle\int_\Omega [\beta(u_{\mathrm{nodal}})]_h v_h \end{cases} = \int_\Omega f v_h.$$

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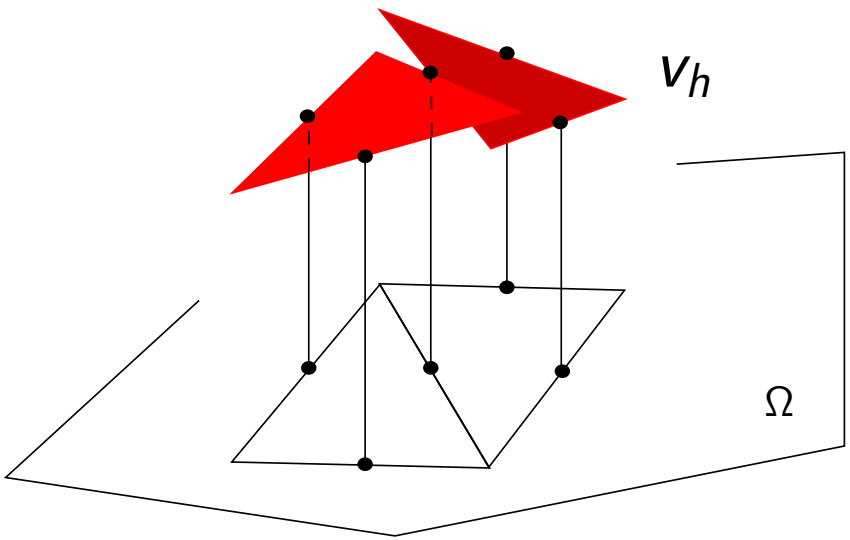
Issues:

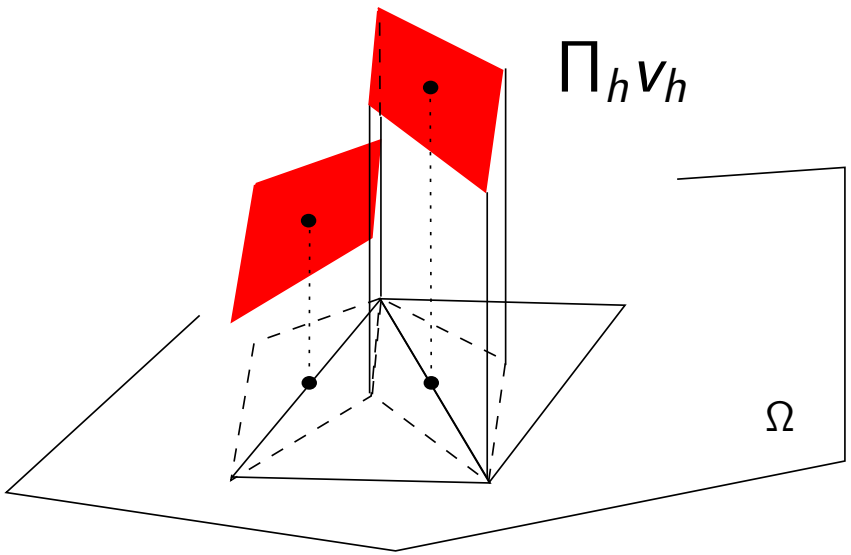
- No exact quadrature for $\int_\Omega \beta(u_h) v_h$.
- No certainty that $\int_\Omega [\beta(u_{\mathrm{nodal}})]_h u_h \geq 0$.

Mass-lumping

Idea: replace functions by piecewise-constant reconstructions:

$$\text{Find } u_h \in E_h \text{ s.t., for all } v_h \in E_h,$$
$$\int_\Omega \nabla_h u_h \cdot \nabla_h v_h + \int_\Omega \beta(\Pi_h u_h)\Pi_h v_h = \int_\Omega f \Pi_h v_h.$$





Plan

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Step 1: select the discrete space and operators

Gradient discretisation: $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$

- $X_{\mathcal{D},0}$ finite dimensional space (encodes the discrete unknowns of the scheme, and accounts for the BCs).
  - *Conforming* $\mathbb{P}_1$*: with* $\mathcal{V}$ *= vertices of the mesh,*

$$X_{\mathcal{D},0} = \{u = (u_{\boldsymbol{v}})_{\boldsymbol{v}\in\mathcal{V}} \,:\, u_{\boldsymbol{v}} = 0 \text{ if } \boldsymbol{v} \in \partial\Omega\}.$$

  - *Non-conforming* $\mathbb{P}_1$*: with* $\mathcal{F}$ *= faces of the mesh,*

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- $v \in X_{\mathcal{D},0} \mapsto \Pi_{\mathcal{D}} v$ linear.
  $\Pi_{\mathcal{D}} v : \Omega \to \mathbb{R}$ "reconstructed function".

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- $v \in X_{\mathcal{D},0} \mapsto \nabla_{\mathcal{D}} v$ linear.
  $\nabla_{\mathcal{D}} v : \Omega \to \mathbb{R}^d$ "reconstructed gradient".
  $\|\nabla_{\mathcal{D}} v\|_{L^p(\Omega)}$ must be a norm on $X_{\mathcal{D},0}$.

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Gradient scheme: in the weak formulation of the PDE, replace the space and operators by the discrete ones coming from $\mathcal{D}$.

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Linear diffusion:

- Weak formulation:

$$\text{Find } \overline{u} \in H_0^1(\Omega) \text{ such that, } \forall \overline{v} \in H_0^1(\Omega),$$
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***p*-Laplace equation**: $-\mathrm{div}(|\nabla \overline{u}|^{p-2}\nabla \overline{u}) = f$

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Plan

3 measures of accuracy

Notation: $\|g\|_{L^p} = \left(\int_\Omega |g|^p\right)^{1/p}$.

Measure of coercivity

$$C_{\mathcal{D}} = \max_{v_{\mathcal{D}} \in X_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}} v_{\mathcal{D}}\|_{L^p}}{\|\nabla_{\mathcal{D}} v_{\mathcal{D}}\|_{L^p}}.$$

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Measure of GD-consistency ("interpolation error" in FE)

$$S_{\mathcal{D}}(\varphi) = \min_{v_{\mathcal{D}} \in X_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v_{\mathcal{D}} - \varphi\|_{L^p} + \|\nabla_{\mathcal{D}} v_{\mathcal{D}} - \nabla\varphi\|_{L^p}\right).$$

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Measure of limit-confirmity ("consistency" in FE)

$$W_{\mathcal{D}}(\boldsymbol{\psi}) = \max_{v_{\mathcal{D}} \in X_{\mathcal{D},0} \setminus \{0\}} \frac{1}{\|\nabla_{\mathcal{D}} v_{\mathcal{D}}\|_{L^p}} \left| \int_\Omega \nabla_{\mathcal{D}} v_{\mathcal{D}} \cdot \boldsymbol{\psi} + \Pi_{\mathcal{D}} v_{\mathcal{D}} \mathrm{div} \boldsymbol{\psi} \right|.$$

Error estimates: Linear anisotropic and heterogeneous diffusion

With $\Lambda(\boldsymbol{x})$ symmetric positive definite matrix, bounded w.r.t. $\boldsymbol{x}$,

- Weak formulation:

$$\text{Find } \overline{u} \in H_0^1(\Omega) \text{ such that, } \forall \overline{v} \in H_0^1(\Omega),$$
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Error estimate:

$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\|_{L^2} + \|\nabla_{\mathcal{D}} u_{\mathcal{D}} - \nabla \overline{u}\|_{L^2} \leq C(1 + C_{\mathcal{D}}) \left[S_{\mathcal{D}}(\overline{u}) + W_{\mathcal{D}}(\nabla \overline{u}) \right].$$

Error estimates: p-Laplace

For $p \in (1, +\infty)$, with $W_0^{1,p}$ generalising H_0^1 to the power p instead of 2 [no longer *Hilbert* space],

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Error estimate: If $p \in (1, 2]$,

$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\|_{L^p} + \|\nabla_{\mathcal{D}} u_{\mathcal{D}} - \nabla \overline{u}\|_{L^p}$$
$$\leq C(1+C_{\mathcal{D}}) \left[S_{\mathcal{D}}(\overline{u}) + S_{\mathcal{D}}(\overline{u})^{p-1} + W_{\mathcal{D}}(|\nabla \overline{u}|^{p-2} \nabla \overline{u}) \right].$$

Error estimates: p-Laplace

For $p \in (1, +\infty)$, with $W_0^{1,p}$ generalising H_0^1 to the power p instead of 2 [no longer *Hilbert* space],

- Weak formulation:

$$\text{Find } \overline{u} \in W_0^{1,p}(\Omega) \text{ such that, } \forall \overline{v} \in W_0^{1,p}(\Omega),$$
$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \overline{v} = \int_\Omega f \overline{v}.$$

- Gradient scheme:

$$\text{Find } u \in X_{\mathcal{D},0} \text{ such that, } \forall v \in X_{\mathcal{D},0},$$
$$\int_\Omega |\nabla_{\mathcal{D}} u|^{p-2} \nabla_{\mathcal{D}} u \cdot \nabla_{\mathcal{D}} v = \int_\Omega f \Pi_{\mathcal{D}} v.$$

Error estimate: If $p \in [2, +\infty)$,

$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\|_{L^p} + \|\nabla_{\mathcal{D}} u_{\mathcal{D}} - \nabla \overline{u}\|_{L^p}$$
$$\leq C(1+C_{\mathcal{D}}) \left[S_{\mathcal{D}}(\overline{u}) + \left[S_{\mathcal{D}}(\overline{u}) + W_{\mathcal{D}}(|\nabla \overline{u}|^{p-2} \nabla \overline{u}) \right]^{\frac{1}{p-1}} \right].$$

Plan

Coercivity, GD-consistency, limit-conformity

- Consider $(\mathcal{D}_m)_{m\in\mathbb{N}}$ sequence of GDs. From the previous estimates,

$$\|\Pi_{\mathcal{D}_m} u_m - \overline{u}\|_{L^p} \to 0 \quad \text{and} \quad \|\nabla_{\mathcal{D}_m} u_m - \nabla\overline{u}\|_{L^p} \to 0$$

provided that the following properties hold.

(P1) Coercivity: $(C_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is bounded.

(P2) GD-consistency: for all $\varphi \in W_0^{1,p}(\Omega)$, $S_{\mathcal{D}_m}(\varphi) \to 0$ as $m \to \infty$.

(P3) Limit-conformity: for all "proper" $\boldsymbol{\psi} : \Omega \to \mathbb{R}^d$, $W_{\mathcal{D}_m}(\boldsymbol{\psi}) \to 0$ as $m \to \infty$.

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- *Actually, (P3)⇒(P1).*

Compactness: for non-linear problems

(P4) Compactness: $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is compact if for all $u_m \in X_{\mathcal{D}_m,0}$ such that $(\|\nabla_{\mathcal{D}_m} u_m\|_{L^p})_{m\in\mathbb{N}}$ is bounded, $(\Pi_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ is relatively compact in L^p.
(Discrete Rellich theorem).

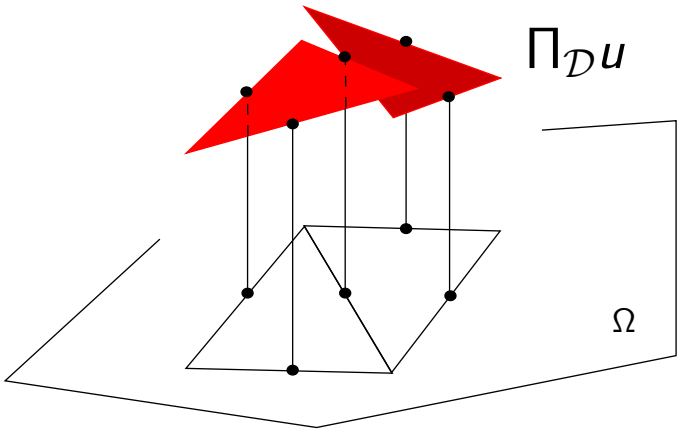
- Useful for $-\mathrm{div}(a(\overline{u})\nabla\overline{u}) = f$ for example.

Piecewise constant reconstruction

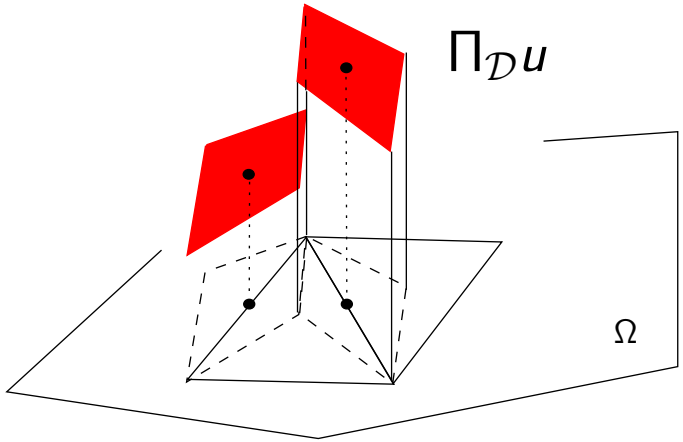
(P5) Piecewise constant reconstruction: $\mathcal{D}$ has a piecewise constant reconstruction if there exists a basis $(e_i)_{i\in I}$ of $X_{\mathcal{D},0}$ and a partition $(\Omega_i)_{i\in I}$ of Ω (some of them can be empty) such that, for all $u = \sum_i u_i e_i \in X_{\mathcal{D},0}$,

$$\Pi_{\mathcal{D}} u = \sum_i u_i \mathbf{1}_{\Omega_i}.$$

- Essential for $-\Delta\overline{u} + \beta(\overline{u}) = f$ (comes from degenerate evolution problems).



Not piecewise-constant



Piecewise-constant

Plan

Conforming Galerkin approximations (e.g. $\mathbb{P}_1$)

Gradient discretisation:

- $X_{\mathcal{D},0}$ finite dimensional subspace of $W_0^{1,p}(\Omega)$,
- $\Pi_{\mathcal{D}} = \mathrm{Id} : X_{\mathcal{D},0} \to W_0^{1,p}(\Omega)$,
- $\nabla_{\mathcal{D}} = \nabla : X_{\mathcal{D},0} \to W_0^{1,p}(\Omega)$.

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Properties:

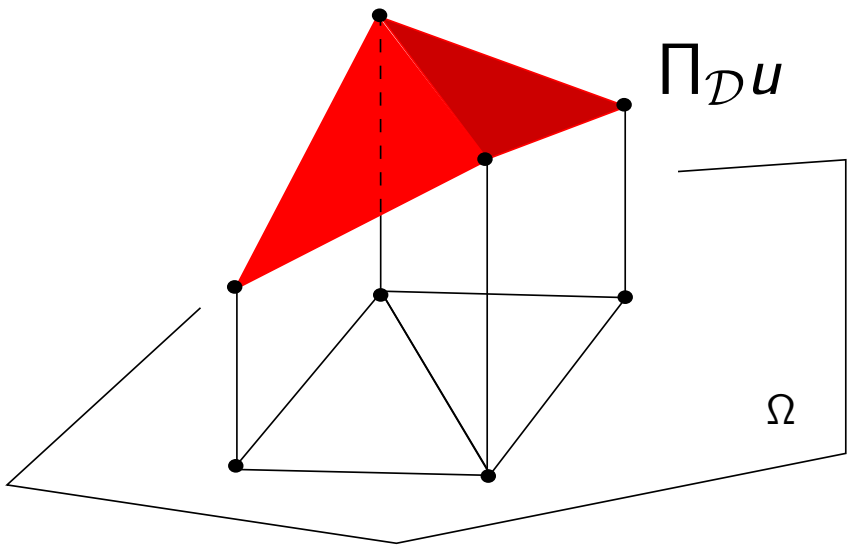
- (P1) Coercivity: $C_{\mathcal{D}} \leq$ Poincaré constant in $W^{1,p}(\Omega)$,
- (P2) GD-consistency: if $\cup_{m\in\mathbb{N}} X_{\mathcal{D}_m,0}$ "ultimately dense" in $W^{1,p}(\Omega)$,
- (P3) Limit-conformity: $W_{\mathcal{D}} \equiv 0$,
- (P4) Compactness: Rellich theorem in $W_0^{1,p}(\Omega)$,
- (P5) Piecewise-constant: No $\rightsquigarrow$ modify into mass-lumped version (no longer conforming).

Conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh.

Gradient discretisation:

- $X_{\mathcal{D},0} = \{u = (u_{\boldsymbol{v}})_{\boldsymbol{v}\text{ vertex}} \,:\, u_{\boldsymbol{v}} = 0 \text{ if } \boldsymbol{v} \in \partial\Omega\}$,
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to W_0^{1,p}(\Omega)$ such that for all $K \in \mathcal{M}$, $(\Pi_{\mathcal{D}} u)_{|K} =$ affine map with value $u_{\boldsymbol{v}}$ at any vertex $\boldsymbol{v}$ of K.
- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to W_0^{1,p}(\Omega)$ such that $(\nabla_{\mathcal{D}} u)_{|K} = \nabla(\Pi_{\mathcal{D}} u)_{|K}$ (broken gradient).

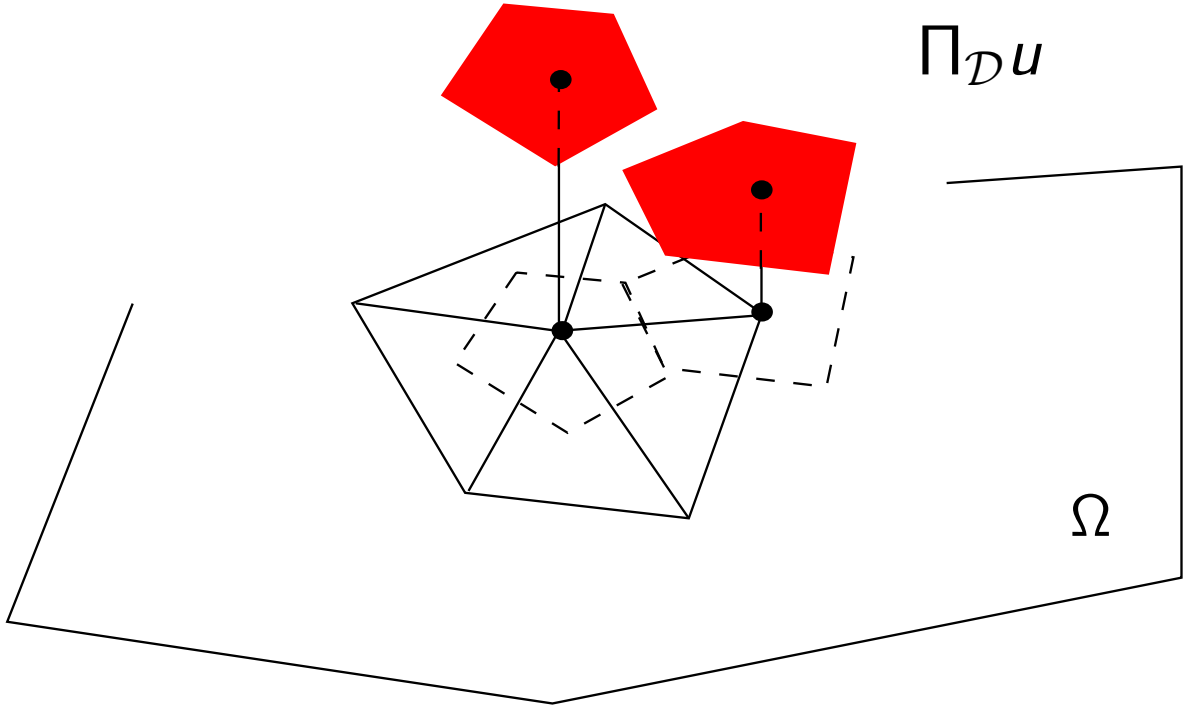


Mass-lumped conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh.

Gradient discretisation:

- $X_{\mathcal{D},0}$, $\nabla_{\mathcal{D}}$ as for $\mathbb{P}_1$ finite elements.
- $\Pi_{\mathcal{D}} u = u_{\boldsymbol{v}}$ on a "dual" cell around $\boldsymbol{v}$.



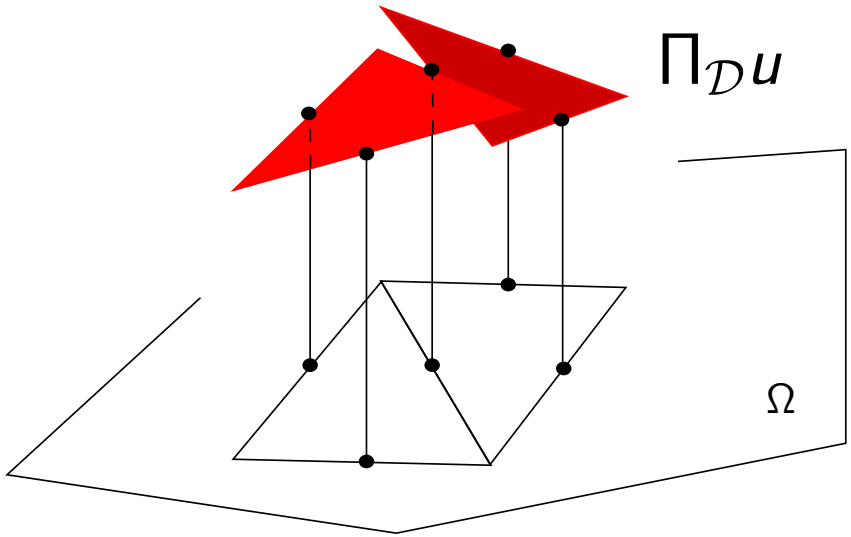
- $(\nabla_{\mathcal{D}} u)_{|K} \neq \nabla(\Pi_{\mathcal{D}} u)_{|K}$.

Non-conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh.

Gradient discretisation:

- $X_{\mathcal{D},0} = \{u = (u_\sigma)_{\sigma\in\mathcal{F}} \,:\, u_\sigma = 0 \text{ if } \sigma \in \mathcal{F}_{\rm ext}\}$,
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to W_0^{1,p}(\Omega)$ such that for all $K \in \mathcal{M}$, $(\Pi_{\mathcal{D}} u)_{|K} =$ affine map with value u_σ at $\overline{\boldsymbol{x}}_\sigma$ for all $\sigma \in \mathcal{F}_K$.
- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to W_0^{1,p}(\Omega)$ such that $(\nabla_{\mathcal{D}} u)_{|K} = \nabla(\Pi_{\mathcal{D}} u)_{|K}$ (broken gradient).

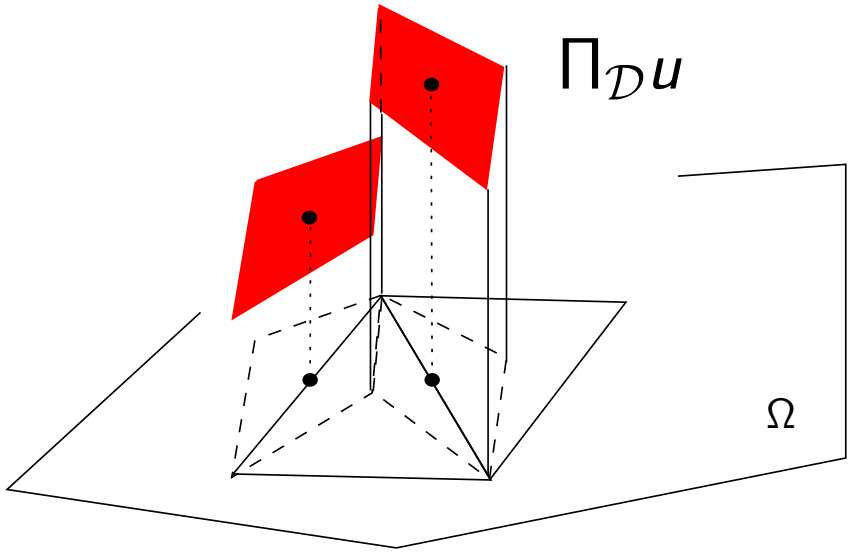


Mass-lumped non-conforming $\mathbb{P}_1$ finite elements

On a triangular/tetrahedral mesh.

Gradient discretisation:

- $X_{\mathcal{D},0}$ and $\nabla_{\mathcal{D}}$ as for non-conforming finite elements,
- $\Pi_{\mathcal{D}} u = u_\sigma$ on a "dual" cell around σ.



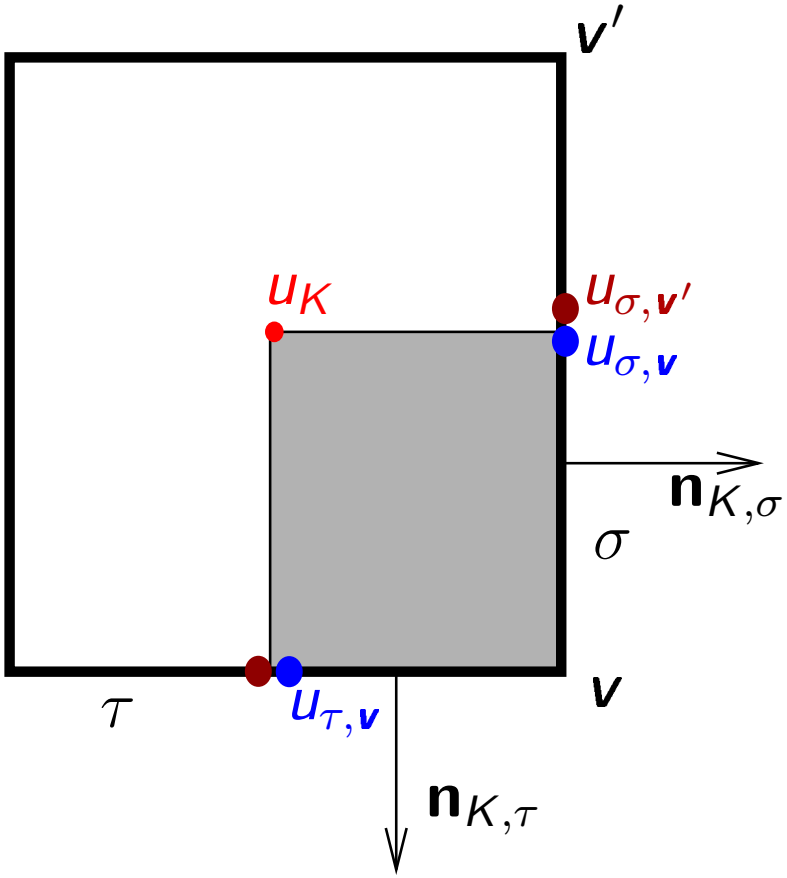
- $(\nabla_{\mathcal{D}} u)_{|K} \neq \nabla(\Pi_{\mathcal{D}} u)_{|K}$.

MPFA-O finite volume scheme

$\mathfrak{T}$=Cartesian mesh (also possible with triangular/tetrahedral).

Gradient discretisation:

- $u \in X_{\mathcal{D},0}$ if $u = ((u_K)_K, (u_{\sigma,\boldsymbol{v}})_{\sigma,\boldsymbol{v}})$ with K cells and $(\sigma, \boldsymbol{v})$ pairs edge-vertex s.t. $\boldsymbol{v} \in \sigma$, and $u_{\sigma,s} = 0$ if $\sigma \in \mathcal{F}_{\text{ext}}$.
- $\Pi_{\mathcal{D}} u = u_K$ in K,
- $\nabla_{\mathcal{D}} u = \dfrac{u_{\sigma,\boldsymbol{v}} - u_K}{\mathrm{d}(x_K, \sigma)} \mathbf{n}_{K,\sigma} + \dfrac{u_{\tau,\boldsymbol{v}} - u_K}{\mathrm{d}(x_K, \tau)} \mathbf{n}_{K,\tau}$ in the cube defined by K and $\boldsymbol{v}$.



Others...

Other methods known to be GDMs:

- Any non-conforming FE method, including non-conforming $\mathbb{P}_k$.
- Mixed finite elements, including $\mathbb{RT}_k$ *(leads to H_{div}-conforming gradient discretisations: $\nabla_{\mathcal{D}} u \in H_{\mathrm{div}}$)*.
- SIPG discontinuous Galerkin method.
- Hybrid Mimetic Mixed methods, including mixed-hybrid Mimetic Finite Differences.
- Nodal Mimetic Finite Difference methods.
- Hybrid high-order methods, non-conforming Virtual Element Methods, non-conforming Mimetic Finite Difference.

Plan

Plan

Tools for mesh-based GDM

Polytopal toolbox

- Provides generic theorems that make the proof of (P1) coercivity, (P3) limit-conformity, and (P4) compactness very easy.

Local linearly exact gradient discretisation:

- Rigorous notion for "$\nabla_{\mathcal{D}}$ exactly reconstructs linear functions" *(nearly all numerical methods try to satisfy this property)*.

- Provides easy proof of (P2) consistency for all methods.

Proof of (P1)–(P4) for non-conforming $\mathbb{P}_1$

Proof of the property $(\mathcal{P})$ *for non-conforming* $\mathbb{P}_1$ *gradient discretisations.* We drop the index m from time to time for sake of legibility, and all constants below do not depend on m or the considered cells/edges. Let us define a control of $\mathcal{D}$ by $\mathcal{T}$ in the sense of Definition 2.29, where $\mathcal{T}$ is the simplicial mesh associated to $\mathcal{D}$, with $\boldsymbol{x}_K = \overline{\boldsymbol{x}}_K = \frac{1}{d+1}\sum_{\sigma\in\mathcal{E}_K}\overline{\boldsymbol{x}}_\sigma$ the centres of gravity of the cells K. We define the linear (injective) mappings $\boldsymbol{\Phi} : X_{\mathcal{D}_m,0} \longrightarrow X_{\mathcal{T}_m,0}$ by $\boldsymbol{\Phi}(u)_K = \frac{1}{d+1}\sum_{\sigma\in\mathcal{E}_K} u_\sigma = \Pi_{\mathcal{D}}u(\boldsymbol{x}_K)$ and $\boldsymbol{\Phi}(u)_\sigma = u_\sigma = \Pi_{\mathcal{D}}u(\overline{\boldsymbol{x}}_\sigma)$.

Since $\boldsymbol{\Phi}(u)_K = \Pi_{\mathcal{D}}u(\boldsymbol{x}_K)$ and $\mathcal{G}_K u = \nabla(\Pi_{\mathcal{D}}u)$ in K, we get

$$\boldsymbol{\Phi}(u)_\sigma - \boldsymbol{\Phi}(u)_K = \mathcal{G}_K u \cdot (\overline{\boldsymbol{x}}_\sigma - \boldsymbol{x}_K). \tag{3.2}$$

Therefore, since $\frac{|\overline{\boldsymbol{x}}_\sigma - \boldsymbol{x}_K|}{d_{K,\sigma}} \le \frac{h_K}{d_{K,\sigma}} \le \theta_{\mathcal{T}}$,

$$\sum_{\sigma\in\mathcal{E}_K} |\sigma| d_{K,\sigma} \left|\frac{\boldsymbol{\Phi}(u)_\sigma - \boldsymbol{\Phi}(u)_K}{d_{K,\sigma}}\right|^p \le \theta_{\mathcal{T}}^p d|K|\,|\mathcal{G}_K u|^p.$$

This implies (2.34). We now observe that the affine function α_σ reaches its extremal values at the vertices of K. It is easy to see that $\alpha_\sigma(\mathsf{v}_\sigma) = 1-d$, where v_σ is the vertex opposite to the face σ, and that $\alpha_\sigma(\mathsf{v}_{\sigma'}) = 1$ for all $\sigma' \neq \sigma$. Therefore, for $\boldsymbol{x} \in K$,

$$|\Pi_{\mathcal{D}}u(\boldsymbol{x}) - \boldsymbol{\Phi}(u)_K| = \left|\sum_{\sigma\in\mathcal{E}_K} (\boldsymbol{\Phi}(u)_\sigma - \boldsymbol{\Phi}(u)_K)\alpha_\sigma(\boldsymbol{x})\right| \le (d+1)\max(1,d-1)\max_{\sigma\in\mathcal{E}_K} |\mathcal{G}_K u \cdot (\overline{\boldsymbol{x}}_\sigma - \boldsymbol{x}_K)|.$$

This inequality implies $\omega^{\Pi}(\mathcal{D},\mathcal{T},\boldsymbol{\Phi}) \le (d+1)\max(1,d-1)h_{\mathcal{M}}$ and therefore (2.35) holds. Finally, recalling that $\Pi_{\mathcal{D}}u$ is affine in each simplex K and that $\nabla_{\mathcal{T}}$ is exact on interpolants of affine functions (cf. Lemma 2.28), we see that $\nabla_{\mathcal{D}}u = \nabla_{\mathcal{T}}\boldsymbol{\Phi}(u)$ in Ω. Hence $\omega^{\nabla}(\mathcal{D},\mathcal{T},\boldsymbol{\Phi}) = 0$ and (2.36) holds. Proposition 2.31 therefore shows that $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is coercive in the sense of Definition 2.2, limit-conforming in the sense of Definition 2.4, and compact in the sense of Definition 2.5.

Since non-conforming $\mathbb{P}_1$ gradient discretisations are LLE gradient discretisations, the consistency of $(\mathcal{D}_m)_{m\in\mathbf{N}}$ follows from Proposition 2.14 by noticing that $\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m)$ is controlled by $\theta_{\mathcal{T}_m}$. □

Proof of (P1)–(P4) for MPFA-O

Proof of the property $(\mathcal{P})$ *for MPFA-O gradient discretisations.* We drop the indices m for sake of legibility. We consider the polytopal mesh $\mathcal{T} = (\mathcal{M}, \mathcal{E}', \mathcal{P}, \mathcal{V}')$ where the sets $(\mathcal{M}, \mathcal{P})$ are those of the original polytopal mesh, $\mathcal{E}' = \{\sigma_{\mathsf{v}}\,;\, \sigma \in \mathcal{E},\ \mathsf{v} \in \mathcal{V}\}$, and $\mathcal{V}'$ is the set of all vertices of the elements of $\mathcal{E}'$. We define a control of $\mathcal{D}$ by $\mathcal{T}$ (in the sense of Definition 2.29) as the isomorphism $\mathbf{\Phi} : X_{\mathcal{D},0} \longrightarrow X_{\mathcal{T},0}$ given by $\mathbf{\Phi}(u)_K = u_K$ and $\mathbf{\Phi}(u)_{\sigma_{\mathsf{v}}} = u_{(\sigma,\mathsf{v})}$. We observe that

$$\int_K |\nabla_{\mathcal{D}} u(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x} \geq C_3 \sum_{\sigma \in \mathcal{E}_K} \sum_{\mathsf{v} \in \mathcal{V}_\sigma} |\sigma_{\mathsf{v}}| d_{K,\sigma} \left| \frac{u_{(\sigma,\mathsf{v})} - u_K}{d_{K,\sigma}} \right|^p,$$

with $C_3 = 1$ for parallelepipedic meshes, and $C_3 > 0$ depends on an upper bound of the regularity of the mesh for simplicial meshes. Therefore $\|\nabla_{\mathcal{D}} u\|^p_{L^p(\Omega)^d} \geq C_3 \|\mathbf{\Phi}(u)\|^p_{\mathcal{T},0,p}$ and (2.34) is proved. Since $\Pi_{\mathcal{D}} u = \Pi_{\mathcal{T}} \mathbf{\Phi}(u)$, we get $\omega^{\Pi}(\mathcal{D}, \mathcal{T}, \mathbf{\Phi}) = 0$, which proves (2.35). Finally, we have

$$\int_K \nabla_{\mathcal{D}} u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \sum_{\sigma \in \mathcal{E}_K} \sum_{\mathsf{v} \in \mathcal{V}_\sigma} |\sigma_{\mathsf{v}}| (u_{\sigma,\mathsf{v}} - u_K) \mathbf{n}_{K,\sigma} = \sum_{\sigma' \in \mathcal{E}'_K} |\sigma'| (\mathbf{\Phi}(u)_{\sigma'} - \mathbf{\Phi}(u)_K) \mathbf{n}_{K,\sigma'} = |K|\ \nabla_{\mathcal{T}} \mathbf{\Phi}(u)_{|K}.$$

This shows that $\omega^{\nabla}(\mathcal{D}, \mathcal{T}, \mathbf{\Phi}) = 0$, which establishes (2.36). Proposition 2.31 therefore shows that $(\mathcal{D}_m)_{m \in \mathbf{N}}$ is coercive in the sense of Definition 2.2, limit-conforming in the sense of Definition 2.4, and compact in the sense of Definition 2.5.

It is proved in [40,41] that the definitions of the approximation points S give the LLE property in both the Cartesian and simplicial cases. Hence, the consistency of $(\mathcal{D}_m)_{m \in \mathbf{N}}$ follows from Proposition 2.14. □

Proof of (P1)–(P4) for HMM

Proof of the property $(\mathcal{P})$ *for HMM gradient discretisations.* Let $(\mathcal{D}_m)_{m\in\mathbf{N}}$ be HMM gradient discretisations built on polytopal meshes $(\mathcal{T}_m)_{m\in\mathbf{N}}$, and let us define a control of $\mathcal{D}_m$ by $\mathcal{T}_m$ in the sense of Definition 2.29. We drop the index m from time to time. Since $X_{\mathcal{D},0} = X_{\mathcal{T},0}$, we can take $\mathbf{\Phi} = \mathrm{Id}$. Estimate (2.34) is given by (3.14). Relation (2.35) follows immediately since $\omega^{\Pi}(\mathcal{D},\mathcal{T},\mathbf{\Phi}) = 0$, owing to $\Pi_{\mathcal{D}}u = \Pi_{\mathcal{T}}u = \Pi_{\mathcal{T}}\mathbf{\Phi}(u)$. Recalling that $|D_{K,\sigma}| = \frac{|\sigma| d_{K,\sigma}}{d}$ we have

$$\int_K \nabla_{\mathcal{D}} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = |K|\nabla_K u + \frac{1}{\sqrt{d}} \sum_{\sigma\in\mathcal{E}_K} |\sigma|[\mathcal{L}_K R_K(Q_K(u))]_\sigma \mathbf{n}_{K,\sigma}. \tag{3.15}$$

The definition of R_K and the property $\sum_{\sigma\in\mathcal{E}_K} |\sigma|\mathbf{n}_{K,\sigma}(\overline{\boldsymbol{x}}_\sigma - \boldsymbol{x}_K)^T = |K|\mathrm{Id}$ (a consequence of Stokes' formula) show that for any $\eta \in \mathrm{Im}(R_K)$ we have $\sum_{\sigma\in\mathcal{E}_K} |\sigma|\eta_\sigma \mathbf{n}_{K,\sigma} = 0$. Hence, since $\mathrm{Im}(\mathcal{L}_K) = \mathrm{Im}(R_K)$, (3.15) gives

$$\int_K \nabla_{\mathcal{D}} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = |K|\nabla_K u = |K|\nabla_{\mathcal{T}}\mathbf{\Phi}(u)_{|K},$$

which shows that $\omega^{\nabla}(\mathcal{D},\mathcal{T},\mathbf{\Phi}) = 0$, and thus that (2.36) holds. The coercivity, limit-conformity and compactness of $(\mathcal{D}_m)_{m\in\mathbf{N}}$ therefore follow from Proposition 2.31. Since HMM gradient discretisations are LLE gradient discretisations, the consistency of $(\mathcal{D}_m)_{m\in\mathbf{N}}$ readily follows from Proposition 2.14, after noticing that the regularity assumption on $(\mathcal{D}_m)_{m\in\mathbf{N}}$ gives a bound on $(\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m))_{m\in\mathbf{N}}$. □

Plan

Convergence analysis by compactness

I. A priori estimates: prove that approximation solutions u_h remain bounded in certain norms.

II. Compactness result: from above estimates, prove convergence up to subsequence of u_h to some $\overline{u}$.

III. Proof that $\overline{u}$ is a solution to the model: "pass to the limit" in the scheme (plug interpolant of smooth test functions) to prove that $\overline{u}$ is a weak solution of the continuous model.

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GDM: provides generic interpolation result for test functions.

Plan

TPFA method for $-\Delta \overline{u} = f$

- Used since the 50' by petroleum engineers.

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- Approximate $\overline{u}$ by u_h piecewise constant on an "admissible" mesh $\mathfrak{T}$.
Classical example: triangular mesh with x_K =circumcenter of K.

- Expected and proved error estimate: $\|u_h - \overline{u}\|_{L^2} = \mathcal{O}(h)$, if $\overline{u}$ smooth enough.

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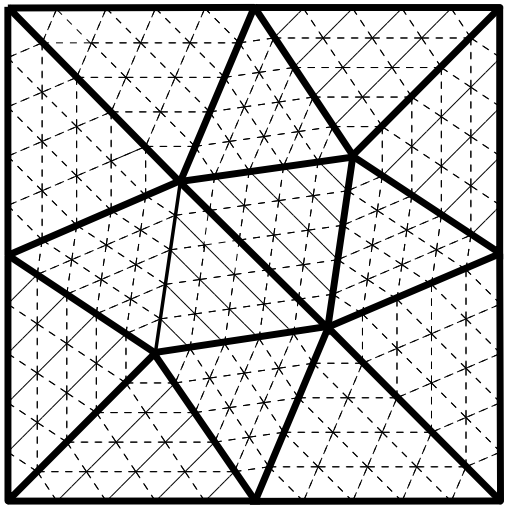
- Experimentally observed: if $P_h\overline{u}$ piecewise constant function equal to $\overline{u}(x_K)$ on $K \in \mathcal{M}$,

$$\boxed{\|u_h - P_h\overline{u}\|_{L^2} = \mathcal{O}(h^2).}$$

Super-convergence result



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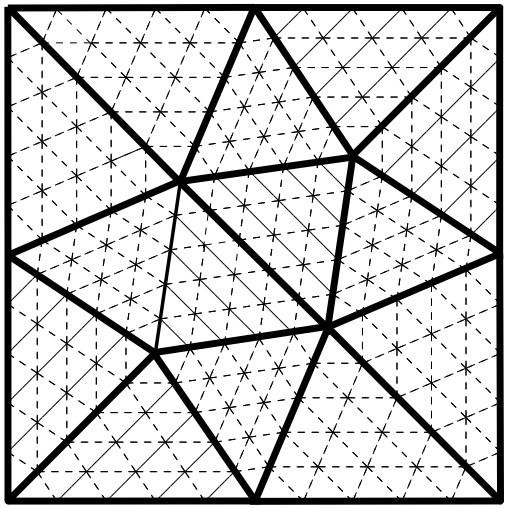
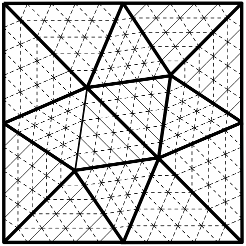
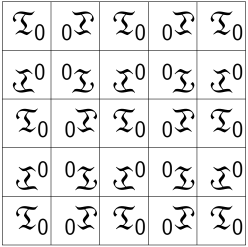
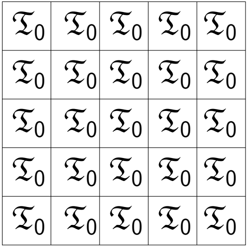


$\mathfrak{T}_0$	${}_0\mathfrak{T}$	$\mathfrak{T}_0$	${}_0\mathfrak{T}$	$\mathfrak{T}_0$
$\mathfrak{T}^0$	${}^0\mathfrak{T}$	$\mathfrak{T}^0$	${}^0\mathfrak{T}$	$\mathfrak{T}^0$
$\mathfrak{T}_0$	${}_0\mathfrak{T}$	$\mathfrak{T}_0$	${}_0\mathfrak{T}$	$\mathfrak{T}_0$
$\mathfrak{T}^0$	${}^0\mathfrak{T}$	$\mathfrak{T}^0$	${}^0\mathfrak{T}$	$\mathfrak{T}^0$
$\mathfrak{T}_0$	${}_0\mathfrak{T}$	$\mathfrak{T}_0$	${}_0\mathfrak{T}$	$\mathfrak{T}_0$

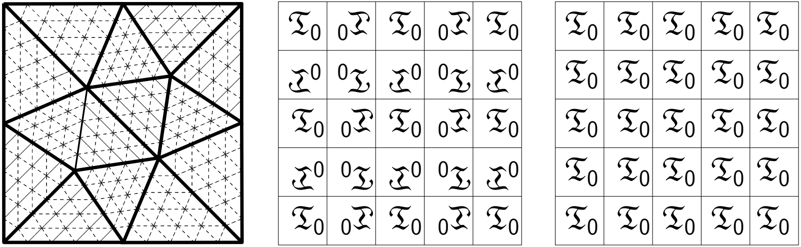
Super-convergence result







Super-convergence result



Theorem (D.–Nataraj; IMA J. Numer. Anal. 2016)

Under H^2 regularity assumption on the PDE, on sequences of triangular meshes as above,

$$\|u_h - P_h\overline{u}\|_{L^2} = \mathcal{O}(h^2).$$

- Very indirect proof (exploits relation HMM–TPFA, skewed projection in L^2, and local compensation properties of the meshes...).

Plan

Model: continuous optimisation problem

Cost functional: $J : \Omega \times \mathbb{R} \times \mathbb{R}^d \to [0, +\infty)$ convex w.r.t. third variable and

$$J(\boldsymbol{x}, s, \xi) \geq \alpha |\xi|^2 - \beta(\boldsymbol{x})$$

Minimisation problem:

Find $\overline{u} \in H_0^1(\Omega)$ that realises $\displaystyle\min_{v \in H_0^1(\Omega)} \int_\Omega J(\boldsymbol{x}, v(\boldsymbol{x}), \nabla v(\boldsymbol{x}))\, \mathrm{d}x.$

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GDM approximation: *the name of the game is substitution...*

Find $u \in X_{\mathcal{D},0}$ that realises $\displaystyle\min_{v \in X_{\mathcal{D},0}} \int_\Omega J(\boldsymbol{x}, \Pi_{\mathcal{D}} v(\boldsymbol{x}), \nabla_{\mathcal{D}} v(\boldsymbol{x}))\,\mathrm{d}x.$

- Covers a wide range of numerical schemes at once...

Convergence by compactness

Take $(\mathcal{D}_m)_{m\in\mathbb{N}}$ that satisfies (P1)–(P4).

A priori estimates: using $0 \in X_{\mathcal{D},0}$,

$$\alpha\|\nabla_{\mathcal{D}_m} u_m\|_{L^2}^2 - \beta \leq \int_\Omega J(\boldsymbol{x}, 0, 0)\,\mathrm{d}\boldsymbol{x}$$

so $(\|\nabla_{\mathcal{D}_m} u_m\|_{L^2})_{m\in\mathbb{N}}$ bounded.

Convergence by compactness

Take $(\mathcal{D}_m)_{m\in\mathbb{N}}$ that satisfies (P1)–(P4).

A priori estimates: $(\|\nabla_{\mathcal{D}_m} u_m\|_{L^2})_{m\in\mathbb{N}}$ bounded.

Compactness results: generic GDM results show that, up to a subsequence,

- $\Pi_{\mathcal{D}_m} u_m \to \overline{u}$ strongly in L^2, with $\overline{u} \in H_0^1(\Omega)$,
- $\nabla_{\mathcal{D}_m} u_m \to \nabla \overline{u}$ weakly in L^2.

Convergence by compactness

Take $(\mathcal{D}_m)_{m\in\mathbb{N}}$ that satisfies (P1)–(P4).

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- $\nabla_{\mathcal{D}_m} u_m \to \nabla \overline{u}$ weakly in L^2.

Limit problem: take $\varphi \in H_0^1(\Omega)$. Generic GDM result give an interpolant $I_{\mathcal{D}_m}\varphi \in X_{\mathcal{D}_m,0}$ such that

$$\Pi_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi \to \varphi \text{ and } \nabla_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi \to \nabla\varphi \text{ strongly in } L^2.$$

Convergence by compactness

Take $(\mathcal{D}_m)_{m\in\mathbb{N}}$ that satisfies (P1)–(P4).

A priori estimates: $(\|\nabla_{\mathcal{D}_m} u_m\|_{L^2})_{m\in\mathbb{N}}$ bounded.

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- $\nabla_{\mathcal{D}_m} u_m \to \nabla \overline{u}$ weakly in L^2.

Limit problem: take $\varphi \in H_0^1(\Omega)$.

$$\Pi_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi \to \varphi \text{ and } \nabla_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi \to \nabla\varphi \text{ strongly in } L^2.$$

By definition of u_m,

$$\int_\Omega J(\boldsymbol{x}, \Pi_{\mathcal{D}_m} u_m, \nabla_{\mathcal{D}_m} u_m)\,\mathrm{d}\boldsymbol{x} \le \int_\Omega J(\boldsymbol{x}, \Pi_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi, \nabla_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi)\,\mathrm{d}\boldsymbol{x}.$$

Convergence by compactness

Take $(\mathcal{D}_m)_{m\in\mathbb{N}}$ that satisfies (P1)–(P4).

A priori estimates: $(\|\nabla_{\mathcal{D}_m} u_m\|_{L^2})_{m\in\mathbb{N}}$ bounded.

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Limit problem: take $\varphi \in H_0^1(\Omega)$.

$$\Pi_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi \to \varphi \text{ and } \nabla_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi \to \nabla\varphi \text{ strongly in } L^2.$$

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$$\int_\Omega J(\boldsymbol{x}, \Pi_{\mathcal{D}_m} u_m, \nabla_{\mathcal{D}_m} u_m)\,\mathrm{d}\boldsymbol{x} \le \int_\Omega J(\boldsymbol{x}, \Pi_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi, \nabla_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi)\,\mathrm{d}\boldsymbol{x}.$$

- RHS: strong convergences of $\Pi_{\mathcal{D}_m} I_{\mathcal{D}_m}$ and $\nabla_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi$.
- LHS: strong convergence of $\Pi_{\mathcal{D}_m} u_m$, weak convergence of $\nabla_{\mathcal{D}_m} u_m$ and convexity of J w.r.t. third variable.

Convergence by compactness

Take $(\mathcal{D}_m)_{m\in\mathbb{N}}$ that satisfies (P1)–(P4).

A priori estimates: $(\|\nabla_{\mathcal{D}_m} u_m\|_{L^2})_{m\in\mathbb{N}}$ bounded.

Compactness results: up to a subsequence,

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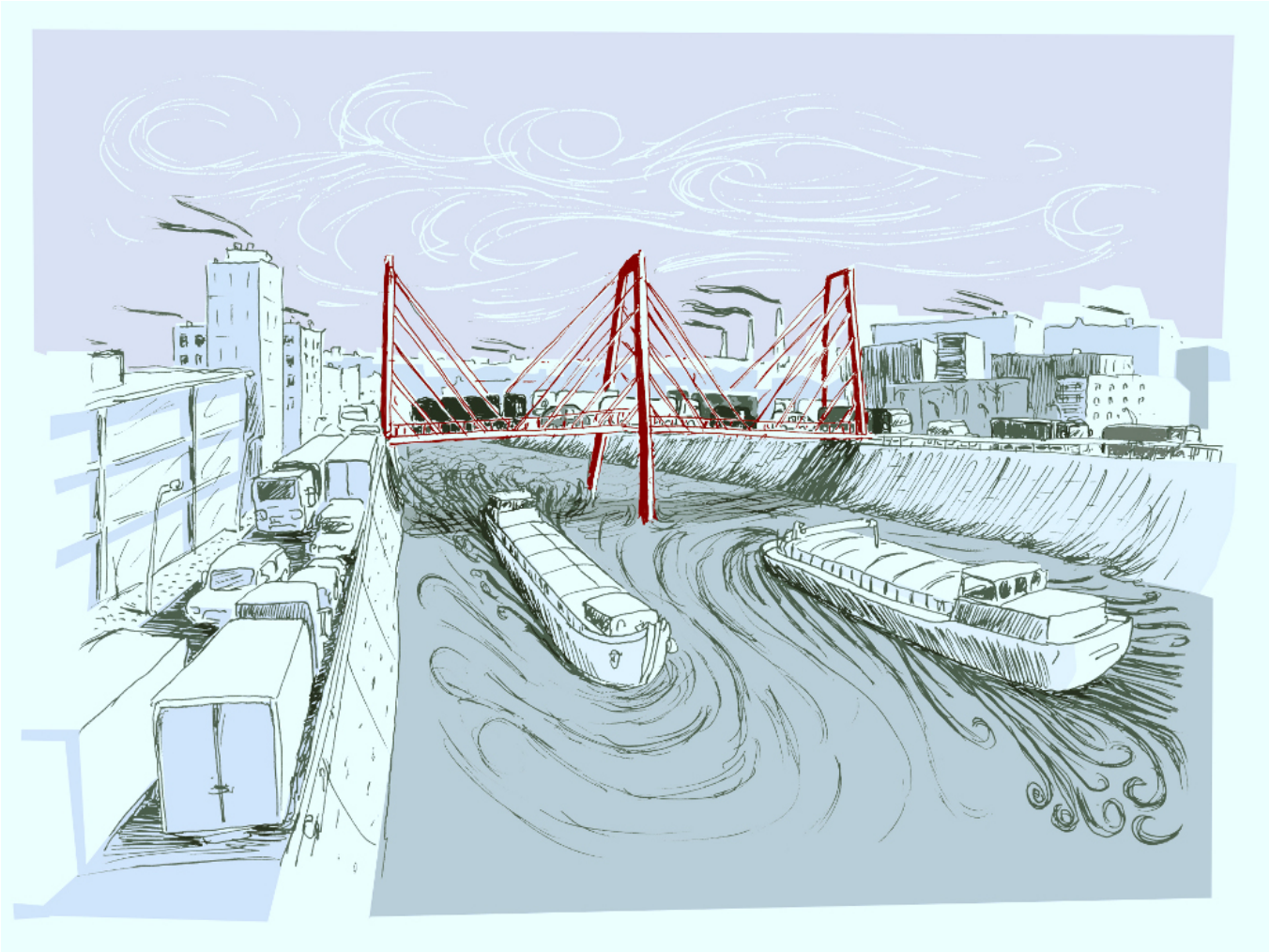
Limit problem: take $\varphi \in H_0^1(\Omega)$.

$$\Pi_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi \to \varphi \text{ and } \nabla_{\mathcal{D}_m} I_{\mathcal{D}_m}\varphi \to \nabla\varphi \text{ strongly in } L^2.$$

$$\boxed{\int_\Omega J(\boldsymbol{x}, \overline{u}, \nabla\overline{u})\,\mathrm{d}\boldsymbol{x} \le \int_\Omega J(\boldsymbol{x}, \varphi, \nabla\varphi)\,\mathrm{d}\boldsymbol{x}.}$$

Conclusion: tools for convergence analysis in real-world situations



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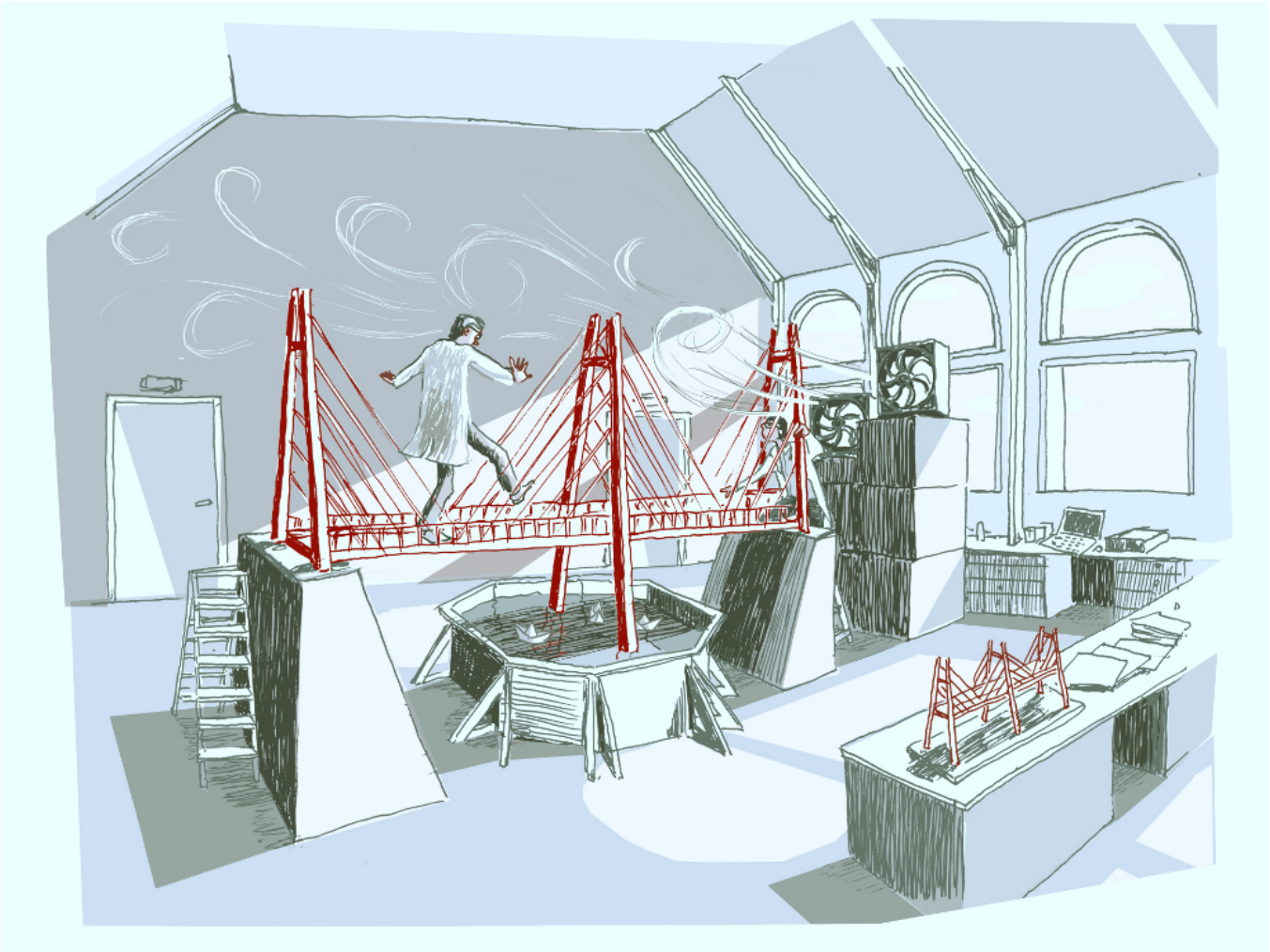
Conclusion: tools for convergence analysis in real-world situations



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- D. and Nataraj – Super-convergence TPFA (IMA JNA, 2017).
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Thanks.