Construction of common fixed points for monotone nonexpansive semigroups

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Abstract

Common Fixed Point for Monotone Nonexpansive Semigroups

We discuss the monotone nonexpansive semigroups of nonlinear mappings acting in Banach spaces equipped with partial order. We provide an algorithm for the construction of a common fixed point for such semigroups and prove its weak convergence. We give examples and interpret our results in the context of the theory of differential equations and dynamical systems.
Let $X$ be a Banach space or metric space. Let $C \subseteq X$ and $T : C \rightarrow C$. We want to find $x \in C$ such that $T(x) = x$.

**Equations**

Every equation $f(x) = 0$ can be re-written as the fixed point problem for the mapping $T(x) = f(x) + x$.

Let $(M, d)$ be a complete metric space and $\varphi : M \rightarrow \mathbb{R}^+$ be l.s.c.

**Optimisation and variational problems (example)**

Ekeland Variational Principle (1974): There exists $v^* \in M$ such that $\varphi(v^*) - \varphi(v) + d(v^*, v) > 0$ for every $v \in M$.

Caristi Fixed Point (1975): If $T : M \rightarrow M$ satisfies $d(x, T(x)) \leq \varphi(x) - \varphi(T(x))$ for every $x \in M$ then $T$ has a fixed point.

Oettli and Thera proved in 1993 that both theorems are equivalent!
Three main streams of FPT

Metric FPT initiated by Banach Fixed Point Theorem (1922)

Let $(M, d)$ be a complete metric space, and $T : M \rightarrow M$ be a contraction, i.e. there exists a constant $\alpha < 1$ such that $d(T(x), T(y)) \leq \alpha d(x, y)$ for all $x, y \in M$. Then, there exists a unique $z \in M$ such that $T(z) = z$. Moreover, for any $x \in M$, there holds $d(T^n(x), z) \rightarrow 0$, where $T^n$ is the $n$-th iterate of $T$.

Order FPT initiated by Knaster-Tarski Theorem (1928, 1955)

Let $(E, \preceq)$ be a complete lattice. Then every order-preserving (monotone) mapping $T : E \rightarrow E$ has a fixed point.

Topological FPT initiated by Brouwer Theorem (1912)

Let $B$ be a closed ball in $\mathbb{R}^n$. Then every continuous mapping $T : B \rightarrow B$ has a fixed point.
Fixed points existence for nonexpansive mappings

Browder (1965)

Let $C \neq \emptyset$ be a closed, convex, bounded subset of a uniformly convex Banach space $X$, and $T : C \to C$ be nonexpansive, i.e.

$$
\|T(x) - T(y)\| \leq \|x - y\| \quad \text{for all } x, y \in C.
$$

Then, there exists a $z \in C$ such that $T(z) = z$. The set of all fixed points is closed and convex.

Browder (1965)

Let $C \neq \emptyset$ be a closed, convex, bounded subset of a uniformly convex Banach space $X$, and $\{T_s\}_{s \in J}$ a commuting family of nonexpansive mappings $T_s : C \to C$. Then, there exists a $z \in C$ such that $T_s(z) = z$ for all $s \in J$. The set of all such common fixed points is closed and convex.
Monotone nonexpansive mappings

Let $X$ be a Banach space endowed with a partial order "$\leq$" such that all order intervals are convex and closed.

Let $C \subset X$ be convex, closed and bounded. We say that $T : C \to C$ is a monotone nonexpansive mapping if it is monotone, $x \leq y \Rightarrow T(x) \leq T(y)$ and

$$\|T(x) - T(y)\| \leq \|x - y\|$$

(1)

for any $x, y \in C$ such that $x$ and $y$ are comparable in the sense of the partial order "$\leq$".

Since (1) needs to hold only for comparable $x$ and $y$, hence $T$ does not have to be nonexpansive or even continuous.
Monotone nonexpansive semigroups

Let \( J \) be a nontrivial subsemigroup of \([0, \infty)\) such that \( 0 \in J \).

**Definition**

A one-parameter family \( S = \{T_s : s \in J\} \) of mappings from \( C \) into itself is said to be a monotone nonexpansive semigroup on \( C \) if \( S \) satisfies the following conditions:

(i) all \( T_s \) are monotone nonexpansive mappings;

(ii) \( T_0(x) = x \) for \( x \in C \);

(iii) \( T_{t+s}(x) = T_t(T_s(x)) \) for \( x \in C \) and \( t, s \in J \);

(iv) for each \( x \in C \), the mapping \( s \mapsto T_s(x) \) is norm continuous;

(v) \( x \preceq T_s(x) \) for \( x \in C \) and \( s \in A \), where \( A \) is generating for \( J \).

We say that \( S \) is equicontinuous if the family of mappings \( \{t \mapsto T_t(x) : x \in C\} \) is equicontinuous at \( t = 0 \).

A set \( A \subset J \) is called a generating set for the parameter semigroup \( J \) if for every \( 0 < u \in J \) there exist \( m \in \mathbb{N} \), \( s \in A \), \( t \in A \) such that \( u = ms + t \).
Bin Dehaish and Khamsi (2016)

Assume that $X$ is uniformly convex in every direction. Let $C \subset X$ be convex, closed and bounded. Let $T : C \to C$ be a monotone nonexpansive mapping. Assume that there exists $x_0 \in C$ such that $x_0$ and $T(x_0)$ are comparable. Then $T$ has a fixed point.

As observed very recently by Espinola and Wisnicki, by using the Knaster-Tarski theorem, this result can be generalized to monotone mappings.

WMK (2017)

Assume $X$ is uniformly convex. Let $S$ be a monotone nonexpansive semigroup on $C$. Assume in addition that there exists $x \in C$ such that $x \preceq T_s(x)$ for all $s \in J$. Then $S$ has a common fixed point $z \in Fix(S)$ such that $x \preceq z$. Moreover, if $f_1, f_2 \in Fix(S)$ are comparable then $f = cf_1 + (1 - c)f_2 \in Fix(S)$ for every $c \in [0, 1]$. 
Example of a monotone nonexpansive mapping

Let $X = L^2([0, 1], \mathbb{R})$ and $B$ be a closed ball in $X$ centered at zero. Define the operator $T : X \to X$ by

$$T(x)(t) = g(t) + \int_0^1 F(t, s, x(s))ds,$$

where $g \in X$ and $F : [0, 1] \times [0, 1] \times X \to \mathbb{R}$ is measurable and satisfies

$$0 \leq F(t, s, x(s)) - F(t, s, y(s)) \leq x(t) - y(t),$$

for $t, s \in [0, 1]$ and $x, y \in X$ such that $y \leq x$ a.e. Assume also that there exists a non-negative function $h(\cdot, \cdot) \in L^2([0, 1] \times [0, 1])$ and $M < \frac{1}{2}$ such that $|F(t, s, u)| \leq h(t, s) + M|u|$, where $t, s, u \in [0, 1]$. Then $T$ is a monotone nonexpansive mapping acting within $B$. Moreover, if $g(t) + \int_0^1 F(t, s, 0)ds \geq 0$ for almost every $t \in [0, 1]$ then $T(0) \geq 0$. Hence, $T$ has a fixed point $z \in B$ and $z \geq 0$. 
Example of a monotone nonexpansive semigroup

Consider the following Initial Value Problem (IVP) for an unknown function $u(x, \cdot) : [0, \infty) \to X$:

$$
\begin{cases}
  u(x, 0) = x \\
  u'(x, t) + (I - H_t)(u(x, t)) = 0,
\end{cases}
$$

where $H_t : C \to C$ are monotone nonexpansive mappings with respect to the order $\preceq$. The notation $u'(x, t)$ denotes the derivative of the function $t \mapsto u(x, t)$. We assume that $x \preceq H_t(x)$ for every $t \in [0, \infty)$. It can be proved that there exists a solution $u(x, \cdot)$ for the (IVP) such that $x \preceq u(x, t)$ for every $t \in [0, \infty)$. Let us denote $T_t(x) = u(x, t)$. It can be proved that $T = \{T_t\}_{t \geq 0}$ is a monotone nonexpansive semigroup. The common fixed points of $T$ can be interpreted as the stationary points of the dynamical process with the evolution function $(t, x) \to T_t(x)$. 
A Banach space $X$ is said to have the Opial property if for each sequence \( \{x_n\} \) of elements of $X$ such that $x_n \rightharpoonup x$, and for any $y \in X$ such that $y \neq x$

\[
\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|.
\]

It is well known that many classical spaces including the Hilbert space and Banach spaces $l^p$ with $1 < p < \infty$ possess the Opial property while such important uniformly convex spaces like $L^p[0, 1]$ ($1 < p < \infty$) fail this condition for $p \neq 2$. In the context of monotone mappings the following concept can provide the remedy.
Monotone Opial property

**Definition**

We say that \((X, \| \cdot \|, \preceq)\) have the monotone Opial property if for each nondecreasing sequence \(\{x_n\}\) of elements of \(X\) such that \(x_n \rightharpoonup x\), and for any \(y \in X\) such that \(y \neq x\) and \(x_n \preceq y\) for every \(n \in \mathbb{N}\)

\[
\liminf_{n \to \infty} \| x_n - x \| < \liminf_{n \to \infty} \| x_n - y \|.
\]

The norm \(\| \cdot \|\) is called monotone if \(u \preceq v \preceq w\) implies that

\[
\max \{\|v - u\|, \|w - v\|\} \leq \|w - u\| \text{ for any } u, v, w \in X.
\]

**Alfuraidan and Khamsi (2017)**

Let \((X, \| \cdot \|, \preceq)\) be a uniformly convex, partially ordered Banach space for which order intervals are convex and closed. Assume that the norm \(\| \cdot \|\) is monotone. Then \(X\) has the monotone Opial property.

As an important example \(L^p[0, 1]\) have the monotone Opial property for any \(p > 1\).
Krasnoselskii-Ishikawa iteration process

**Definition**

Given \( \lambda \in (0, 1) \) and a sequence of parameters \( t_n \in J \). We will consider the following iteration process, often called in the literature the Krasnoselskii-Ishikawa iteration process (KIS)

\[
x_{n+1} = (1 - \lambda)x_n + \lambda T_{t_n}(x_n),
\]

where \( x_0 \in C \) is the starting element chosen so that \( x_0 \preceq T_t(x_0) \) for \( t \in J \). It can be proved that \{\( x_n \)\} is nondecreasing.

Also, \( C_{x_0} = C \cap [x_0, \rightarrow) \) is nonempty, weakly compact and \( T_s(C_{x_0}) \subset C_{x_0} \) for \( s \in J \).

Set \( K = \bigcap_{n=0}^{\infty} C \cap [x_n, \rightarrow) \). Observe that \( K \neq \emptyset \) is weakly compact and that \( T_s(K) \subset K \) for \( s \in J \).
Theorem (WMK 2017)

Let $X$ be uniformly convex with monotone norm (hence with monotone Opial property as well). Let $S = \{T_t : t \in J\}$ be an equicontinuous monotone nonexpansive semigroup of mappings on $C \subset X$ where $C$ is closed, convex and bounded. Choose $\{t_k\} \subset J$ so that for every $s \in J$ there exists a strictly increasing sequence $\{j_k\}$ such that $\{j_{k+1} - j_k\}$ is bounded and $t_{j_{k+1}} = s + t_{j_k}$ for every $k \in \mathbb{N}$. Let $x_0 \in C$ be such that $x_0 \preceq T_s(x_0)$ for every $s \in E$, $\overline{E} = A$. Let $\{x_k\}$ be a sequence given by $(KIS)$. Then $x_k \rightharpoonup w$, where $w$ is a common fixed point for $S$. Moreover, $x_0 \preceq w$ and $w$ is a minimal common fixed point with this property.

Typical examples of Banach spaces for which our Theorem can be applied are: $l^p$ and $L^p[0, 1]$ for $p > 1$; Orlicz spaces $L^\varphi$ with the Luxemburg norm if $\varphi$ is strictly convex and satisfies the condition $\Delta_2$, modular function spaces with the $\Delta_2$ and $(UUC2)$. 

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Monotone nonexpansive semigroups
Key results needed for the proof the Convergence Theorem

From now on $X$ is assumed to be uniformly convex.

**Lemma 1 (WMK 2017)**

Let $\{x_k\}$ be $(KIS)$ sequence. Assume that for every $s \in E$ there exists a strictly increasing, quasi-periodic sequence $\{j_k : k = 1, 2, \ldots \}$ such that $t_{j_{k+1}} = s + t_{j_k}$. Then $\lim_{k \to \infty} \|T_s(x_k) - x_k\| = 0$ for every $s \in J$.

**Lemma 2 (WMK 2017)**

Assume that the norm in $X$ is monotone. Let $\{x_k\}$ be $(KIS)$ sequence. If $x_{n_k} \rightharpoonup w$ and for every $s \in J$, $\|T_s(x_{n_k}) - x_{n_k}\| \to 0$ then $w \in Fix(S)$.

**Lemma 3 (WMK 2017)**

If $w \in Fix(S)$ is such that $x_0 \preceq w$ then $w \in K$ and there exists $r \in \mathbb{R}$ such that $\lim_{k \to \infty} \|x_k - w\| = r$. 

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Monotone nonexpansive semigroups
Sketch of the proof of the Convergence Theorem

Restrict to $C_{x_0}$. By Lemma 1, $\lim_{k \to \infty} \|T_s(x_k) - x_k\| = 0$ for every $s \in J$. Consider two weak cluster points of $\{x_k\}$, $y$ and $z$. There exist then two subsequences $\{y_k\}$ and $\{z_k\}$ of $\{x_k\}$ such that $y_k \rightharpoonup y$ and $z_k \rightharpoonup z$. By Lemma 2, $y, z \in Fix(S)$. By Lemma 3 the following limits exist

$$r_1 = \lim_{k \to \infty} \|x_k - y\|, \quad r_2 = \lim_{k \to \infty} \|x_k - z\|,$$

and $y, z \in K$. Claim: $y = z$. Assume to the contrary that $y \neq z$. Note $x_k \preceq y$ and $x_k \preceq z$ because $y, z \in K$. By monotone Opial property

$$r_1 = \liminf_{k \to \infty} \|y_k - y\| < \liminf_{k \to \infty} \|y_k - z\| = r_2$$

$$= \liminf_{k \to \infty} \|z_k - z\| < \liminf_{k \to \infty} \|z_k - y\| = r_1.$$

The contradiction implies $y = z$, which means that $\{x_k\}$ has at most one weak cluster point in weakly-compact $C_{x_0}$, hence $x_k \rightharpoonup w$. Applying Lemma 2 again, we conclude that $w \in Fix(S)$. Since $w \in K$, $x_0 \preceq w$. 

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Monotone nonexpansive semigroups

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Sketch of the proof of the Convergence Theorem, cont.

It remains to be proved that \( w \) is a minimal common fixed point greater than or equal to \( x_0 \). To this end let us assume to the contrary that there exists \( z \in \text{Fix}(S) \) such that \( x_0 \preceq z \) and \( z \prec w \). By Lemma 3, \( x_k \preceq z \) for \( k = 1, 2, \ldots \). Using the monotone Opial property we obtain that

\[
\liminf_{n \to \infty} \| x_n - w \| < \liminf_{n \to \infty} \| x_n - z \|
\]

On the other hand, since the norm is monotone, we conclude from \( x_n \preceq z \prec w \) that

\[
\| x_n - z \| \leq \| x_n - w \|.
\]

The contradiction completes the proof.
Example of a construction of a sequence \( \{ t_k \} \)

Let us order the elements from \( E \) so that \( 0 = \alpha_1 < \alpha_2 < \alpha_3 < \ldots \).

We will consider segments, that is, finite sequences of the real numbers of the form \( B = \{ \beta_1, \beta_2, \ldots, \beta_m \} \). Given a real number \( r \) we define

\[ rB = \{ r\beta_1, r\beta_2, \ldots, r\beta_m \}. \]

Let us define three initial segments:

\[ B_1 = \{ \alpha_1, \alpha_2 \}, \]
\[ B_2 = 2\{ \alpha_1, \alpha_2, \alpha_1, \alpha_3 \}, \]
\[ B_3 = 3\{ \alpha_1, \alpha_2, \alpha_1, \alpha_3, \alpha_1, \alpha_2, \alpha_1, \alpha_4 \}. \]

Generally, given a segment \( B_n \) we define \( B_{n+1} = (n + 1)D_{n+1} \), where \( D_{n+1} \) is constructed as a concatenation of \( B_n \) to itself and replacing the last element by \( \alpha_{n+1} \). Concatenation of all segments \( B_1, B_2, \ldots \) will give a sequence \( \{ t_k \} \) with required properties.

Note: that \( t_n \to \infty \).
Let $X$ be uniformly convex with monotone norm (hence with monotone Opial property as well). Let $S = \{T_t : t \in J\}$ be an equicontinuous monotone nonexpansive semigroup of mappings on $C \subset X$ where $C$ is closed, convex and bounded. Choose $\{t_k\} \subset J$ so that for every $s \in J$ there exists a strictly increasing sequence $\{j_k\}$ such that $\{j_{k+1} - j_k\}$ is bounded and $t_{j_{k+1}} = s + t_{j_k}$ for every $k \in \mathbb{N}$. Let $x_0 \in C$ be such that $x_0 \preceq T_s(x_0)$ for every $s \in E$, $\overline{E} = A$. Let $\{x_k\}$ be a sequence given by $(KIS)$. Then $x_k \rightharpoonup w$, where $w$ is a common fixed point for $S$. Moreover, $x_0 \preceq w$ and $w$ is a minimal common fixed point with this property.

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