Probability

- Probability
- Tree diagrams
- Lattice diagrams
- Venn diagrams
- Karnough maps
- Probability tables
- Union & intersection rules
- Conditional probability
- Markov chains
Probability

- Probability is the mathematics of chance.
- While an outcome of a random event can only be can’t determined in advance, we can use a prediction based on experimental or theoretical probability.
- Probabilities can be expressed as fractions, decimals or percentages.
- $0 = \text{no chance at all}, \ 100\% = \text{absolutely certain}.$
- The chances of an event happening or not happening add to make 1.

\[
\begin{align*}
0\% & \text{ - no chance} \\
25\% & \text{ - unlikely} \\
50\% & \text{ - even chance} \\
75\% & \text{ - likely} \\
100\% & \text{ - certain}
\end{align*}
\]

\[
O \leq \Pr (E) \leq 1
\]

\[
\Pr (E') = 1 - \Pr (E)
\]
• **Experimental probability**: based on measurement of past outcomes. For example, a 15% chance of a wet grand final day (based on the last 100 years).

• **Theoretical probability**: based on the number of favourable outcomes from the total set of possible outcomes. For example, a 1 in a 1000 chance of buying the winning ticket in a raffle.

• The long term *expected value* is found by multiplying the probability & number of trials.

• For example, buying 20 game tickets with a 1 in 10 chance, it is expected that 2 prizes would be won.

**Experimental probability** = \[
\frac{\text{Number of favourable outcomes observed}}{\text{Total number of trials}}
\]

**Theoretical probability** = \[
\frac{\text{Number of possible favourable outcomes}}{\text{Total number of possible outcomes}}
\]

\[
\Pr(E) = \frac{n(E)}{n(\varepsilon)}
\]

**Expected value** = \[
E(x) = n \times Pr(x)
\]
Calculating probability - tree diagrams

- Shows all possible outcomes for sequential events.
- Each branch on the tree represents a possible outcome.
- For example, tossing three coins:
- Each outcome has a 1 in 8 chance \( \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \).

Pr (3 heads) = \( \frac{1}{8} \)
Pr (2 heads) = \( 3 \times \frac{1}{8} = \frac{3}{8} \)
Pr (1 head) = \( 3 \times \frac{1}{8} = \frac{3}{8} \)
Pr (0 heads) = \( \frac{1}{8} \)
Calculating probability - lattice diagrams

- Shows the possible outcomes from two sequential events.
- Easier to draw than a tree diagram if there is a large number of outcomes.
- For example, rolling two dice & finding the product.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tr>
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<td>6</td>
<td>12</td>
<td>18</td>
<td>24</td>
<td>30</td>
<td>36</td>
</tr>
</tbody>
</table>

The chance of scoring 6:

\[
Pr(6) = 4 \times \frac{1}{36} = \frac{1}{9}
\]

The chance of scoring 36:

\[
Pr(36) = 1 \times \frac{1}{36} = \frac{1}{36}
\]
Venn diagrams

- A useful visual tool to show the relationship between two sets or events.

25 students in the class
- 10 competed in the swimming sports (Set S)
- 14 competed in the athletics sports (Set A)
- 8 students competed in both (Set S ∩ A)

(S ∩ A) = the intersection of events S & A.

- Two events are mutually exclusive if there is no intersection of the two sets.
- (If no students did both sports)
Union & intersection rules

What is the chance that a student that is randomly selected:

a) Is a swimmer?  
\[ \text{Pr}(S) = \frac{10}{25} = 40\% \]

b) Is an athlete?  
\[ \text{Pr} = \frac{14}{25} = 56\% \]

c) Is a swimmer and an athlete?  
\[ \text{Pr} = \frac{8}{25} = 32\% \] (Intersection)

d) Is a swimmer or an athlete?  
\[ \text{Pr} = \frac{10}{25} + \frac{14}{25} - \frac{8}{25} = \frac{16}{25} = 64\% \] (Union)

\[
\text{Pr}(A \cup B) = \text{Pr}(A) + \text{Pr}(B) - \text{Pr}(A \cap B)
\]

\[
\text{Pr}(A \cup B) = 1 - \text{Pr}(A' \cap B')
\]
Karnough maps

- Used to show the number of elements of two overlapping sets.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>A’</th>
<th></th>
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<tbody>
<tr>
<td>S</td>
<td>8</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>S’</td>
<td>6</td>
<td>9</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>11</td>
<td>25</td>
</tr>
</tbody>
</table>

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Karnough maps

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>B'</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(n(A \cap B))</td>
<td>(n(A \cap B'))</td>
<td>(n(A))</td>
</tr>
<tr>
<td>A'</td>
<td>(n(A' \cap B))</td>
<td>(n(A' \cap B'))</td>
<td>(n(A'))</td>
</tr>
<tr>
<td></td>
<td>(n(B))</td>
<td>(n(B'))</td>
<td>(n)</td>
</tr>
</tbody>
</table>
Probability tables

- Used to show the probability of two overlapping events occurring.

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>B'</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Pr(A ∩ B)</td>
<td>Pr(A ∩ B')</td>
<td>Pr(A)</td>
</tr>
<tr>
<td>A'</td>
<td>Pr(A' ∩ B)</td>
<td>Pr(A' ∩ B')</td>
<td>Pr(A')</td>
</tr>
<tr>
<td></td>
<td>Pr(B)</td>
<td>Pr(B')</td>
<td>1</td>
</tr>
</tbody>
</table>
Conditional probability

- Two events (A & B) are considered to be independent if the probability of event A occurring has no influence over event B occurring.
- If the events A & B are not independent, then they are said to be conditional.

\[
Pr(A \cap B) = Pr(A) \times Pr(B)
\]

Independent events:

\[
Pr(A \cap B) = Pr(A \mid B) \times Pr(B)
\]

Conditional events:

\[
Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)}
\]

- For example, a student in the class that competed at the athletics - what is the chance that they were also at the swimming sports?

\[
Pr(S \mid A) = \frac{Pr(S \cap A)}{Pr(A)} = \frac{\frac{8}{25}}{\frac{14}{25}} = \frac{8}{14} = \frac{4}{7}
\]
Markov chains

- For example, the chance of a team winning is 80% following a win, but only 40% following a loss.
- What is the chance that after winning the first game, the team also wins the third?
- This could be expressed as a tree diagram......

\[
\begin{align*}
\Pr(W_2 \cap W_3) &= 80\% \times 80\% = 64\% \\
\Pr(W_2 \cap W_3') &= \Pr(W_3' | W_2) \times \Pr(W_2) \\
\Pr(W_2' \cap W_3) &= 80\% \times 20\% = 16\% \\
\Pr(W_2' \cap W_3') &= \Pr(W_3 | W_2') \times \Pr(W_2') \\
\Pr(W_2' \cap W_3) &= 20\% \times 40\% = 8\% \\
\Pr(W_2' \cap W_3') &= \Pr(W_3 | W_2') \times \Pr(W_2') \\
\Pr(W_3) &= \Pr(W_2 \cap W_3) + \Pr(W_2' \cap W_3) = 64\% + 8\% = 72\% 
\end{align*}
\]
Markov chains

- Used to find the long term probability of a sequence of a number of repeated events, where the conditional probabilities remain constant.

\[
\begin{align*}
\Pr(B) &= \Pr(B \mid A) \times \Pr(A) + \Pr(B \mid A') \times \Pr(A') \\
\begin{bmatrix}
\Pr(B) \\
\Pr(B')
\end{bmatrix} &= 
\begin{bmatrix}
\Pr(B \mid A) & \Pr(B \mid A') \\
\Pr(B' \mid A) & \Pr(B' \mid A')
\end{bmatrix}^n 
\begin{bmatrix}
\Pr(A) \\
\Pr(A')
\end{bmatrix}
\end{align*}
\]

- \(S_n = (\text{Result})\) 
- \(T = \text{Transition matrix (probabilities)}\) 
- \(S_0 = \text{State matrix (initial state)}\) 

\[S_n = T^n \times S_0\]
Markov chains

• From the previous example:

The first game was a win

\[
\begin{bmatrix}
0.8 & 0.4 \\
0.2 & 0.6 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
\end{bmatrix}

= \begin{bmatrix}
0.72 \\
0.28 \\
\end{bmatrix}
\]

S\_2 = State (Result) matrix (after 2 more games)

T = Transition matrix (probabilities)

S\_0 = State matrix (initial state)

• As the number of states is increased, the overall probabilities will converge to defined limits.

\[n = 1, Pr = 0.800\]
\[n = 2, Pr = 0.720\]
\[n = 3, Pr = 0.688\]
\[n = 4, Pr = 0.675\]
\[n = 15, Pr = 0.667\]
\[n = 100, Pr = 0.667\]

Overall chance of winning

\[= 0.4 / (0.4 + 0.2)\]
\[= \frac{2}{3}\]