Four-Valued First-Order Semantics for RW

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These slides: `https://tinyurl.com/ShaysMelbourneTalk`

October 19, 2017

The plan:

- I'm going to spell out a semantic theory for you.
- I'm going to tell you why you should like it.
- Then I'm going to go to bed.

More detail on the first two points:

- Step 1: I'm going to remind you who RWQ is.
- Step 2: I'll then give you a semantic theory for the zero-order fragment of RWQ.
- Step 3: Then I'll give a varying-domain stratified semantic theory for full RWQ.
- Step 4: I'll raise the standard objections to the stratified semantics.
- Step 5: I'll respond to the objections, along the way tossing out a constant-domain stratified semantic theory.

Note: At no point will we go through soundness or completeness proofs. They're just too messy for this sort of talk.

But send me an email if you want to read the results! When the paper is in a readable state I'll pass it on.

The Languages

Zero-Order Polyadic sentential language.

- Predicates (of any arity)
- Names
- Connectives: $\wedge$, $\neg$, $\rightarrow$.
- $\alpha \vee \beta =_{def} \neg(\neg\alpha \wedge \neg\beta)$.

First-Order Add $\forall$ and variables.

- $\exists x =_{def} \neg\forall x \neg$

Define wff and sentence in the usual ways.

The Logic (part I)

A1 $\alpha \to \alpha$

A2 $(\alpha \wedge \beta) \to \alpha$

A3 $(\alpha \wedge \beta) \to \beta$

A4 $((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to (\alpha \to (\beta \wedge \gamma))$

A5 $(\alpha \wedge (\beta \vee \gamma)) \to ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$

A6 $\neg\neg\alpha \to \alpha$

A7 $(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$

A8 $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

A9 $\alpha \to ((\alpha \to \beta) \to \beta)$

The Logic (part II)

R1 $\dfrac{\alpha, \alpha \to \beta}{\beta}$

R2 $\dfrac{\alpha, \beta}{\alpha \wedge \beta}$

Zero-order RW is the logic generated by A1-A9, R1-R2, restricted to the zero-order language.

The Logic (part III)

A10 $\forall\nu\phi \to \phi(\tau/\nu)$ (τ free for ν in ϕ).

A11 $\forall\nu(\phi \to \psi) \to (\phi \to \forall\nu\psi)$ (ν not free in ϕ).

A12 $\forall\nu(\phi \vee \psi) \to (\phi \vee \forall\nu\psi)$ (ν not free in ϕ).

R3 $\dfrac{\phi}{\forall\nu\phi}$

Logic generated by all A1-A12 and R1-R3 is RWQ.

(This is Ross Brady's axiomatization)

Zero-Order Semantics (part I)

A zero-order premodel is a 7-tuple $\langle D, S, N, R, \delta, \mathcal{E}^+, \mathcal{E}^- \rangle$ where

- D is a set (the *domain*).
- S is a set (*setups*).
- $N \subseteq S$ (*normal* setups).
- $R \subseteq S^3$ (compatibility).
- δ is a *denotation function*.
- $\mathcal{E}^+(P, a)$ is the *extension* of P at a.
- $\mathcal{E}^-(P, a)$ is the *antiextension* of P at a.

We say a premodel is *reduced* when $N = \{g\}$.

Defintions

- $Rabcd =_{def}$ for some x $Rabx$ and $Rxcd$
- $Ra(bc)d =_{def}$ for some x, $Rbcx$ and $Raxd$.
- $x \leq y =_{def}$ for some $n \in N$, $Rnxy$.

Zero-Order Semantics (part II)

A zero-order model is a zero-order premodel such that

Ordering For all s, t, and u in S

- $s \leq s$.
- If $s \leq t$ and $t \leq u$, then $s \leq u$.
- If $s \leq t$ and $t \leq s$, then $s = t$.

Monotonicity For all a, b, c, and x in S,

- If $a \leq x$ and $Rxbc$, then $Rabc$.
- If $b \leq x$ and $Raxc$, then $Rabc$.
- If $x \leq c$ and $Rabx$, then $Rabc$.

Closure If $n \in N$ and $n \leq m$, then $m \in N$.

Rearranging If $Rabcd$, then

- **B :** $Ra(bc)d$
- **B':** $Rb(ac)d$
- **C :** $Racbd$

Horizontal Atomic Heredity If P is an i-ary predicate and $a \leq b$, then

If $\langle d_1, \ldots, d_i \rangle \in \mathcal{E}^{\pm}(P, a)$, then $\langle d_1, \ldots, d_i \rangle \in \mathcal{E}^{\pm}(P, b)$.

Truth values are in $\{\emptyset, \{1\}, \{0\}, \{1, 0\}\}$:

- $1 \in M^a(P\alpha_1 \ldots \alpha_n)$ iff $\langle \delta(\alpha_1), \ldots, \delta(\alpha_n) \rangle \in \mathcal{E}^+(P, a)$
- $0 \in M^a(P\alpha_1 \ldots \alpha_n)$ iff $\langle \delta(\alpha_1), \ldots, \delta(\alpha_n) \rangle \in \mathcal{E}^-(P, a)$
- $1 \in M^a(\phi \wedge \psi)$ iff $1 \in M^a(\phi)$ and $1 \in M^a(\psi)$.
- $0 \in M^a(\phi \wedge \psi)$ iff $0 \in M^a(\phi)$ or $0 \in M^a(\psi)$.
- $1 \in M^a(\neg\phi)$ iff $0 \in M^a(\phi)$
- $0 \in M^a(\neg\phi)$ iff $1 \in M^a(\phi)$
- $1 \in M^a(\phi \rightarrow \psi)$ iff for all b and c, if $Rabc$ then
  - If $1 \in M^b(\phi)$, then $1 \in M^c(\psi)$, and
  - If $0 \in M^b(\psi)$, then $0 \in M^c(\phi)$.
- $0 \in M^a(\phi \rightarrow \psi)$ iff for some b and c with $Rbca$, $1 \in M^b(\phi)$ and $0 \in M^c(\psi)$.

ϕ is zero-order valid when for all models M, if n is normal, then $1 \in M^n(\phi)$

Theorem

If ϕ is a zero-order theorem, then ϕ is zero-order valid.

Theorem

If ϕ is zero-order valid, then ϕ is a zero-order theorem.

Stratified Models: Generatlities

Some history:

- In two papers in the late 1980s, Kit Fine showed
  - (a) Incompleteness wrt the naïve first-order models.
  - (b) Completeness wrt *stratified* first-order models.
- Impetus for stratified models comes from Fine's prior work on arbitrary objects.

Stratified Models: Generalities

Here's a recipe for building a stratified model:

1. Stack up a family of zero-order models.
2. Single out an increasing sequence of sets of 'arbitrary' objects.
3. Ensure the arbitrary objects behave like arbitrary objects.
4. Use the arbitrary objects to define truth for quantified sentences.
5. Call the resulting stack of zero-order models a first-order model.

"...a universal sentence $\forall x \psi(x)$ is true just in case $\psi(x)$ is true of an arbitrary or generic individual. But let me not be misunderstood. My saying that $\psi(x)$ is true of an arbitrary individual is not a fancy way of saying that $\psi(x)$ is true of every individual. I mean to be taken literally; for the universal sentence $\forall x \psi(x)$ to be true, there must actually be an arbitrary individual of which the condition $\psi(x)$ is true" (Fine, 1988)

Varying-Domain RWQ Premodels

In our case, here's what this looks like:
Varying-domain RWQ-premodels are 5-tuples $\langle D, \Omega, \delta, \mathcal{M}, \Downarrow \rangle$

- D is a set (the base domain)
- $\Omega = \{\omega_i\}_{i=1}^{\infty}$ is a set that is disjoint from D. (arbitrary objects)
- δ is a denotation function.
- $\mathcal{M}$ is a function mapping each finite set X of natural numbers to a zero-order-model M_X.
- $\Downarrow$ is a family of *restriction functions*

Requirements: if $M_X = \langle D_X, S_X, N_X, R_X, \delta, \mathcal{E}^+_X, \mathcal{E}^-_X \rangle$, then

- $D_X = D \cup \{\omega_i\}_{i \in X}$.
- If $X \supseteq Y$, there is a function $\downarrow^X_Y : S_X \to S_Y$ in $\Downarrow$. (Write $\downarrow^X_Y$ with postfix notation)
- For $a \in S_X$, $a{\downarrow^X_Y}{\downarrow^Y_Z} = a{\downarrow^X_Z}$
- $\downarrow^X_X = id_{S_X}$

Terminology: Let $a \in S_X$ and $m, n \in D_X$.

- We say that a is **symmetric** in m and n exactly when
- a does not **extensionally distinguish** m from n; that is, when
- $\langle d_1, \ldots, m, \ldots, d_i \rangle \in \mathcal{E}^{\pm}_X(P, a)$ iff $\langle d_1, \ldots, n, \ldots, d_i \rangle \in \mathcal{E}^{\pm}_X(P, a)$

Varying-Domain RWQ Models

Varying-domain RWQ-models are varying-domain RWQ-premodels that satisfy the following six conditions:

Heredity If $a{\downarrow}^X_Y = b$, then $\mathcal{E}^{\pm}_X(P, a) \cap D^i_Y = \mathcal{E}^{\pm}_Y(P, b)$.

Normality $a{\downarrow}^X_Y \in N_Y$ iff $a \in N_X$.

Lifting If $a \in S_X$, $b \in S_Y$, and $a{\downarrow}^X_{X\cap Y} = b{\downarrow}^Y_{X\cap Y}$ then for some $c \in S_{X\cup Y}$, $a = c{\downarrow}^{X\cup Y}_X$ and $b \leq c{\downarrow}^{X\cup Y}_Y$.

Homomorphism If a, b, and c are in S_X and $R_X abc$, then $R_Y a{\downarrow}^X_Y b{\downarrow}^X_Y c{\downarrow}^X_Y$.

Extension If a, b, and c are in S_Y and $R_Y abc$, then

- if $d{\downarrow}^X_Y = a$ then there are e and f such that $e{\downarrow}^X_Y = b$, $f{\downarrow}^X_Y = c$ and $R_X def$; and
- if $f{\downarrow}^X_Y = c$ then there are d and e such that $d{\downarrow}^X_Y = a$, $e{\downarrow}^X_Y = b$ and $R_X def$.

Symmetry If $a \in S_Y$, $X \supseteq Y$, $m \in D_Y$ and $n \in D_X - D_Y$, then there is a b that is symmetric in m and n such that $b{\downarrow}^X_Y = a$.

Variable Assignments

If ν is a variable and X is a finite set of numbers then a variable assignment maps the *pair* $\langle \nu, X \rangle$ to an element of D_X.

va is X-*coherent* when for all ν and $Y \supseteq X$, $\text{va}(\nu, X) = \text{va}(\nu, Y)$.

Notice if va is X-coherent, then va is Y-coherent when $Y \supseteq X$.

Variants

For ν is a variable and $d \in D \cup \Omega$ we define va^{ν}_{d} as follows:

$$\mathsf{va}^{\nu}_{d}(\chi, X) = \begin{cases} \mathsf{va}(\chi, X) & \text{if } \chi \neq \nu \text{ or } d \notin D_X \\ d & \text{if } \chi = \nu \text{ and } d \in D_X \end{cases}$$

Notice: if va is X-coherent then va^{ν}_{d} is Y-coherent if and only if $D_X \cup \{d\} \subseteq D_Y$.

Last bits:

If τ is a term then by $\varepsilon_X^{\mathsf{va}}(\tau)$ we mean whichever of $\delta(\tau)$ and $\mathsf{va}(\tau, X)$ is appropriate.

Now we can define truth!

How truth works

Let $M = \langle D, \Omega, \delta, \mathcal{M}, \Downarrow \rangle$ be an RWQ-model.

Let X be a finite set of numbers.

Let $a \in S_X$.

Let va be an X-coherent variable assignment.

Let ϕ be an RWQ-wff.

Then $M^a_X(\text{va}, \phi)$ is defined as follows:

- $1 \in M^a_X(\text{va}, P\tau_1 \ldots \tau_n)$ iff $\langle \varepsilon^{\text{va}}_X(\tau_1), \ldots, \varepsilon^{\text{va}}_X(\tau_n) \rangle \in \mathcal{E}^+_X(P, a)$
- $0 \in M^a_X(\text{va}, P\tau_1 \ldots \tau_n)$ iff $\langle \varepsilon^{\text{va}}_X(\tau_1), \ldots, \varepsilon^{\text{va}}_X(\tau_n) \rangle \in \mathcal{E}^-_X(P, a)$
- $1 \in M^a_X(\text{va}, \phi \wedge \psi)$ iff $1 \in M^a_X(\text{va}, \phi)$ and $1 \in M^a_X(\text{va}, \psi)$.
- $0 \in M^a_X(\text{va}, \phi \wedge \psi)$ iff $0 \in M^a_X(\text{va}, \phi)$ or $0 \in M^a_X(\text{va}, \psi)$.
- $1 \in M^a_X(\text{va}, \neg\phi)$ iff $0 \in M^a_X(\text{va}, \phi)$
- $0 \in M^a_X(\text{va}, \neg\phi)$ iff $1 \in M^a_X(\text{va}, \phi)$

  $\vdots$

- $1 \in M_X^a(\mathrm{va}, \phi \to \psi)$ iff for all b and c, if $Rabc$ then
  If $1 \in M_X^b(\mathrm{va}, \phi)$ then $1 \in M_X^c(\mathrm{va}, \psi)$, and
  If $0 \in M_X^b(\mathrm{va}, \psi)$ then $0 \in M_X^c(\mathrm{va}, \phi)$.
- $0 \in M_X^a(\mathrm{va}, \phi \to \psi)$ iff for some b and c with $Rbca$, $1 \in M_X^b(\mathrm{va}, \phi)$ and $0 \in M_X^c(\mathrm{va}, \psi)$.
- $1 \in M_X^a(\mathrm{va}, \forall \nu \phi)$ iff for some $Y \supsetneq X$ and $i \in Y - X$, for all $b \in S_Y$, if $b{\downarrow}_X^Y = a$, then $1 \in M_Y^b(\mathrm{va}_{\omega_i}^{\nu}, \phi)$.
- $0 \in M_X^a(\mathrm{va}, \forall \nu \phi)$ iff for every $Y \supsetneq X$ and $i \in Y - X$ there is a $b \in S_Y$ such that $b{\downarrow}_X^Y = a$ and $0 \in M_Y^b(\mathrm{va}_{\omega_i}^{\nu}, \phi)$.

Note that this succeeds in recursively specifying a function because in the last two clauses the assumptions guarantee that $\mathrm{va}^{\nu}_{\omega_i}$ will be Y-coherent.

M makes ϕ true when if va is X-coherent and $n \in N_X$, then $1 \in M^n_X(\mathrm{va}, \phi)$.

ϕ is V-valid when ϕ is true in every varying-domain RWQ-model.

Theorem

If ϕ is an RWQ-theorem, then ϕ is V-valid.

Theorem

If ϕ is V-valid, then ϕ is an RWQ-theorem.

Taking Stock

We've seen two semantic theories now: the zero-order theory and the varying-domain first-order theory.

I'm going to give you one more.

(It won't be that bad, I promise!)

But first, let's look at what some folks have had to say about the varying-domain theory.

The Complaints

Two main complaints about stratified semantic theories generally:

1. They're varying domain and that's weird!
2. They're really complex and that's weird!

More intelligent versions of these complaints have been voiced by Ross Brady:

'It is understandable that for quantified modal logics that possible worlds might have differing domains from world to world, but this is not clear for practical non-modalized examples such as Peano arithmetic. Indeed, logical applications generally have fixed domains of objects, such as natural numbers or sets, and one should not have to vary such a domain when replacing classical logic by a supposedly superior logic.' [Brady, 2017]

‘[My] general concern with complexity is as follows. Put oneself in the mind of a reasoner conducting a simple inference step and ask the question: what is the rationale or justification for the inference? . . . [A] reasoner is not going to embrace much complexity in making and justifying a single inference step. The logic governing the step would be clean and clear, based on well-understood concepts.’ [Brady 2017]

Complexity

My response to the complexity issue has three pieces:

- Given Fine's aims, the complexity was unavoidable.
- Once we look at stratified models for *particular* logics, much of the complexity goes away.
- Even if it didn't, it's not clear it would be a problem.

On the first point:

Fine gave us a semantic theory that, with mild tweaks, could capture a range of logics between BQ and RQ.

So his base theory had to be adaptable to each of the pathologies of these different logics.

Second point: the theory I've presented is just not that complex.

The zero-order models are fairly orthodox.

The stratified models are just stacks of zero-order models that satisfy the six constraints I gave.

Finally, I'll confess that I don't see the complexity as an issue in any event.

First, there's no reason to expect that an inference that is *semantically complex* will in general be *cognitively demanding*.

Even if it did, I see no reason to expect all logically valid inferences to be cognitively simple.

Varying Domains

My response to the worry about varying domains is this:

There's nothing inherently 'varying-domain' about stratified models.

That is, we can build non-varying-domain stratified models.

Of course, this isn't much of a response if all the successful applications of the stratified semantics require varying domains.

The Lede

But they don't.

We can build a constant-domain version of the above semantics and (I think) *still* get RWQ.

The key idea: stash the arbitrary objects in the '$\emptyset$' truth value to keep them out of trouble.

Constant Domain Stratified Models

Just as in the varying-domain case, a constant-domain stratified model is a 5-tuple $\langle D, \Omega, \delta, \mathcal{M}, \Downarrow \rangle$.

Most things are as before.

There are only three interesting changes required:

Change 1

For the varying-domain models, we required that Ω and D be disjoint.

Now we require that $\Omega = \{\omega_i\}_{i=1}^{\infty} \subseteq D$.

We let $\mathcal{D} = D - \Omega$. δ is now a function from names to $\mathcal{D}$.

And we require that the domains be constant: so for every X, $M_X = \langle D, S_X, N_X, R_X, \delta, \mathcal{E}_X^+, \mathcal{E}_X^- \rangle$.

Change 2

We adopt a new *featurelessness* condition:

Featurelessness If $a \in S_X$, $i \notin X$ and $\langle d_1, \ldots, d_n \rangle \in \mathcal{E}^+_X(P, a) \cup \mathcal{E}^-_X(P, a)$, then none of the d_j's are ω_i.

So in the 'Xth' strata, the arbitrary objects ω_i for $i \notin X$ are featureless – they enter into neither the extension nor the antiextension of any predicate.

(This is what I meant when I said we could 'stash' the extra arbitrary objects in the $\emptyset$ truth-value.)

Change 3

The final change is to make variable assignments simpler: they are now just functions from the variables into D.

Along with this we can simplify things and use the usual definition of x-variants.

What Doesn't Change

Importantly, we keep the old definition of truth for quantified sentences:

- $1 \in M^a_X(\mathrm{va}, \forall\nu\phi)$ iff for some $Y \supsetneq X$ and $i \in Y - X$, for all $b \in S_Y$, if $b{\downarrow}^Y_X = a$, then $1 \in M^b_Y(\mathrm{va}^{\nu}_{\omega_i}, \phi)$.
- $0 \in M^a_X(\mathrm{va}, \forall\nu\phi)$ iff for every $Y \supsetneq X$ and $i \in Y - X$ there is a $b \in S_Y$ such that $b{\downarrow}^Y_X = a$ and $0 \in M^b_Y(\mathrm{va}^{\nu}_{\omega_i}, \phi)$.

So the stratification still matters – it's what lets us get the semantics for the quantifiers 'right'.

But the domains don't vary.

M makes ϕ true when for any va and $n \in N_X$, $1 \in M_X^n(\text{va}, \phi)$.

ϕ is C-valid when ϕ is true in every constant-domain RWQ-model.

Theorem

If ϕ is an RWQ-theorem, then ϕ is C-valid.

Conjecture

If ϕ is C-valid, then ϕ is an RWQ-theorem.

Thanks!

- These slides:
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