Substructural Logics with Fixed Points

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Fixpoints in logic and computer science

Usually:

- Knaster-Tarski (or Banach)

  Given a complete lattice L, any monotone map $f : L \longrightarrow L$ has a fixpoint.

- least/greatest fixpoints for monotone formulas
- foundation of induction/coinduction

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Today:

- Brouwer

  Any continuous map $f : [0,1]^n \longrightarrow [0,1]^n$ has a fixpoint.

- fixpoints for arbitrary formulas
- related to naive comprehension

Systems to be discussed

	Fixpoints	Naive set theory
Łukasiewicz logic	$Ł_{fix}$	$Ł_{set}$
Monoidal t-norm logic	$\mathbf{MTL}_{fix}$	$\mathbf{MTL}_{set}$
Int. logic without contraction	$\mathbf{FLew}_{fix}$	$\mathbf{FLew}_{set}$

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$\mathbf{Int}$ is inconsistent with fixpoints

$\mathbf{Int}$ (intuitionistic logic) with self-contradiction

$$(sc) \quad \alpha \leftrightarrow \neg\alpha$$

is inconsistent (α a propositional constant).

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- Cut elimination procedure works stepwise, though does not terminate.
- Induction on the cut formula is not available.

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FLew: **Int** without contraction.
= intuitionistic multiplicative-additive linear logic + weakening

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□ Proofs shrink by reducing (principal) cuts:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \neg\alpha\end{array}}{\Gamma \Rightarrow \alpha} \quad \dfrac{\begin{array}{c}\vdots\\ \neg\alpha \Rightarrow \Pi\end{array}}{\alpha \Rightarrow \Pi}}{\Gamma \Rightarrow \Pi} \quad \longrightarrow \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \neg\alpha\end{array} \quad \begin{array}{c}\vdots\\ \neg\alpha \Rightarrow \Pi\end{array}}{\Gamma \Rightarrow \Pi}$$

even though the cut formula may become more complicated.

Rule (wc_n):

$$\frac{\overbrace{\Gamma, \ldots, \Gamma}^{n+1}, \Sigma \Rightarrow}{\underbrace{\Gamma, \ldots, \Gamma}_{n}, \Sigma \Rightarrow}\ (wc_n)$$

Weaker forms of contraction

Rule (wc_n):

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Rule (c'):

$$\frac{\Gamma, \Gamma, \Sigma \Rightarrow \Pi \quad \Delta, \Delta, \Sigma \Rightarrow \Pi}{\Gamma, \Delta, \Sigma \Rightarrow \Pi}\ (c')$$

Both admit stepwise cut elimination procedures.
Do they terminate?

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Fact

1. $\mathbf{FLew} + (wc_n)$ is inconsistent with $\beta \leftrightarrow \neg\beta^n$.
2. $\mathbf{FLew} + (c')$ is consistent with any fixpoints.

System $\mathbf{FLew}_{fix}$

Connectives: $\wedge$, $\vee$, $\cdot$, $\rightarrow$, 1, 0

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We also consider mutual fixpoints: $\alpha = A(\beta),\ \beta = B(\alpha)$

$$\alpha \;=\; A(B(\alpha)) \;=\; A(B(A(\beta))) \;=\; A(B(A(B(\alpha)))) \;=\; \cdots$$

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More generally we assume: given n formulas in n variables $A_1(\vec{p}), \ldots, A_n(\vec{p})$, there are constants $\alpha_1, \ldots, \alpha_n$ such that

$$\begin{array}{rcl} \alpha_1 & = & A_1(\alpha_1, \ldots, \alpha_n) \\ \vdots & & \vdots \\ \alpha_n & = & A_n(\alpha_1, \ldots, \alpha_n) \end{array}$$

This defines System $\mathbf{FLew}_{fix}$.

Consistency of $\mathbf{FLew}_{fix}$ + (c')

Fact

$\mathbf{FLew}_{fix} + (c')$ is consistent.

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Proof Idea: any proof of contradiction shrinks by stepwise cut elimination.

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Compare $|d_A|$ and $|d_B|$. If $|d_A| \leq |d_B|$,
the left proof is smaller than the original one.

Actually we have a more general result. Note that $\mathbf{FLew} + (c')$ is a sublogic of Ł (blackboard).

Theorem

Let $\mathbf{L}$ be an axiomatic extension of $\mathbf{FLew}$.

1. If $\mathbf{L}$ is above $\mathbf{FLew} + (wc_n)$ for some n, $\mathbf{L}_{fix}$ is inconsistent.
2. If $\mathbf{L}$ is below Ł, $\mathbf{L}_{fix}$ is consistent.

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Problem 1

Sharpen the above theorem.

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- $\mathbf{FLew}_{set}$ is a basis for resource bounded set theory.

Terms and formulas:

$$
\begin{array}{rcl}
t & ::= & x \mid \{x : \varphi\} \\
\varphi & ::= & t \in t \mid 0 \mid \varphi \to \varphi \mid \forall x.\varphi
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$\mathbf{FLew}_{set}$: extension of $\mathbf{FLew}\forall$ with naive comprehension:

$$t \in \{x : \varphi(x)\} \quad \leftrightarrow \quad \varphi(t).$$

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Eg. let $R := \{x : x \notin x\}$. Then

$$R \in R \quad \leftrightarrow \quad R \notin R.$$

More generally, any set $\{A_1(\vec{p}), \ldots, A_n(\vec{p})\}$ admits a mutual fixpoint.

Hence $\mathbf{FLew}_{fix}$ is embeddable into $\mathbf{FLew}_{set}$.

Consistency of $\mathbf{FLew}_{set}$

There is a shrinking cut elimination procedure:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow \varphi(t)}}{\Gamma \Rightarrow t \in \{x : \varphi(x)\}} \quad \dfrac{\dfrac{\vdots}{\varphi(t) \Rightarrow \Pi}}{t \in \{x : \varphi(x)\} \Rightarrow \Pi}}{\Gamma \Rightarrow \Pi} \quad \longrightarrow \quad \dfrac{\dfrac{\vdots}{\Gamma \Rightarrow \varphi(t)} \quad \dfrac{\vdots}{\varphi(t) \Rightarrow \Pi}}{\Gamma \Rightarrow \Pi}$$

Theorem (Grisin 1982)

$\mathbf{FLew}_{set}$ is consistent.

Expressivity of $\mathbf{FLew}_{set}$

We may define

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This allows us to define a term $\mathbb{N}$ such that

$$x \in \mathbb{N} \quad \leftrightarrow \quad x = 0 \vee \exists y \in \mathbb{N}.\, x = y + 1.$$

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Theorem

Provability in $\mathbf{FLew}_{set}$ is Σ^0_1-complete.

However, $\mathbf{FLew}_{set}$ is a very weak theory, which is analogous to Robinson's $\mathbf{Q}$ in arithmetic.

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In naive set theory, we extend $\mathbf{FLew}_{set}$ with controlled contractions.

Modally controlled contraction 1

We may extend $\mathbf{FLew}_{set}$ with K-modality !:

$$\frac{\Gamma \Rightarrow B}{!\Gamma \Rightarrow !B} \qquad \frac{!A, !A, \Gamma \Rightarrow \Pi}{!A, \Gamma \Rightarrow \Pi}$$

This is called the elementary affine set theory.

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$$\mathbf{N} \;:=\; \{x : \forall X.\, !\forall y(y \in X \to y{+}1 \in X) \to !(0 \in X \to x \in X)\}$$

It supports elementary induction principle:

$$\frac{A(0) \quad \forall y.A(y) \to A(y+1)}{\forall x \in \mathbf{N}.!A(x)}$$

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Theorem (Girard 98, T. 04)

A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is elementary recursive iff it is provably total in elementary affine set theory.

We may also extend $\mathbf{FLew}_{set}$ with two modalities $!, \S$ with

$$\frac{A \Rightarrow B}{!A \Rightarrow !B} \qquad \frac{!A, !A, \Gamma \Rightarrow \Pi}{!A, \Gamma \Rightarrow \Pi} \qquad \frac{\Gamma, \Delta \Rightarrow B}{!\Gamma, \S\Delta \Rightarrow \S B}$$

This is called the light affine set theory.

Modally controlled contraction 2

We may also extend $\mathbf{FLew}_{set}$ with two modalities $!, \S$ with

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This is called the light affine set theory.

Theorem (Girard 98, T. 04)

A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is polynomial time computable iff it is provably total in light affine set theory.

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Summary

- $\mathbf{FLew}_{set}$ is a very weak naive set theory.
- It can be extended by modality ! controlling contraction.
- If ! is K, it captures elementary recursive functions.
- If ! is functorial and bounded by K-modality §, it captures polynomial time functions.
- If ! is T, it is inconsistent.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!\alpha \Rightarrow !\alpha}{!\alpha, \neg !\alpha \Rightarrow}}{!\alpha, \alpha \Rightarrow}}{!\alpha, !\alpha \Rightarrow}\,(T)}{!\alpha \Rightarrow}\,(c)}{\Rightarrow \neg !\alpha}}{\Rightarrow \alpha} \qquad (\alpha = \neg !\alpha)$$

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$$\frac{!\alpha \Rightarrow !\alpha}{!\alpha, \neg !\alpha \Rightarrow}$$

Problem 2

Is K4 consistent? What about other modalities?

$$\frac{\dfrac{!\alpha \Rightarrow}{\Rightarrow \neg !\alpha}}{\Rightarrow \alpha} \qquad (\alpha = \neg !\alpha)$$

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Łukasiewicz's infinite-valued logic

$$\text{Ł} \;:=\; \mathbf{FLew} +$$

(axiom Ł) $\quad ((A \to B) \to B) \to (B \to A) \to A$

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$$\text{(axiom Ł)} \qquad ((A \to B) \to B) \to (B \to A) \to A$$

It allows us to define

$$A \vee B \; := \; (A \to B) \to B$$

$$\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, A \to B \Rightarrow B}}{A \Rightarrow (A \to B) \to B} \qquad \dfrac{\dfrac{\dfrac{B \Rightarrow B \quad A \Rightarrow A}{B, B \to A \Rightarrow A}}{B \Rightarrow (B \to A) \to A}}{B \Rightarrow (A \to B) \to B} \;(\text{Ł})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow C}{C \to B \Rightarrow A \to B}}{(A \to B) \to B, C \to B \Rightarrow B}}{(A \to B) \to B \Rightarrow (C \to B) \to B} \quad \dfrac{\dfrac{B \Rightarrow C}{(B \to C) \to C \Rightarrow C}}{(C \to B) \to B \Rightarrow C} \;(\text{Ł})}{(A \to B) \to B \Rightarrow C}$$

Łukasiewicz's infinite-valued logic

(axiom Ł) $((A \to B) \to B) \to (B \to A) \to A$

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For linear logicians: Ł is an extension of $\mathbf{MLL}$ in which addtives are multiplicatively definable.

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Theorem (Kowalski 2012)

Let A, B be $\to$-only formulas.

- If $(A \to B) \to B$ is provable in $\mathbf{FLew}$, either A or B is provable.
- The following inference is admissible in $\mathbf{FLew}$

$$\frac{\Rightarrow (A \to B) \to B}{\Rightarrow (B \to A) \to A}$$

Łukasiewicz interpretation

Łukasiewicz and Tarski (1930) assigned to each formula

$$B \;\equiv\; B(\beta_1, \ldots, \beta_n)$$

a function

$$[\![B]\!] : [0,1]^n \longrightarrow [0,1]$$

defined by

$$\begin{array}{lcl} [\![\beta_i]\!](\vec{x}) & := & x_i \\ [\![0]\!](\vec{x}) & := & 0 \\ [\![B \to C]\!](\vec{x}) & := & \min(1,\ 1 - [\![B]\!](\vec{x}) + [\![C]\!](\vec{x})) \\ [\![B \cdot C]\!](\vec{x}) & := & \max(0,\ [\![B]\!](\vec{x}) + [\![C]\!](\vec{x}) - 1) \end{array}$$

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Theorem

This is the only assignment on $[0,1]$ which is both $\mathbf{FLew}$-sound and continuous.

Brower's fixpoint theorem $\Rightarrow$ Consistency of Ł$_{fix}$

Theorem (Brouwer 1910)

Every continuous map $f : [0,1]^n \longrightarrow [0,1]^n$ has a fixed point.

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Every continuous map $f : [0,1]^n \longrightarrow [0,1]^n$ has a fixed point.

Corollary

Ł$_{fix}$ is consistent.

Given $A_1(\vec{\alpha}), \ldots, A_n(\vec{\alpha})$, consider

$$([\![A_1]\!], \ldots, [\![A_n]\!]) : [0,1]^n \longrightarrow [0,1]^n$$

and let $(r_1, \ldots, r_n)$ be a fixed point.

Then valuation $v(\alpha_i) := r_i$ satisfies all $\alpha_i \leftrightarrow A_i(\vec{\alpha_i})$. Hence Ł$_{fix}$ is consistent.

Towards a proof theory of $Ł_{fix}$

Two reasons to study proof theory of $Ł_{fix}$:

1. Con($Ł_{fix}$) implies BFT.
2. First step to the consistency of $Ł_{set}$, which is a big open problem in fuzzy logic.

Two reasons to study proof theory of Ł$_{fix}$:

1. Con(Ł$_{fix}$) implies BFT.
2. First step to the consistency of Ł$_{set}$, which is a big open problem in fuzzy logic.

Note: White (1979) introduced a natural deduction system for Ł$_{set}$ and "proved" its consistency. It has been believed correct until recently. But it turned out incorrect (look at a note on my webpage).

McNaughton functions

A McNaughton function is a continuous piecewise-linear function $f : [0,1]^n \longrightarrow [0,1]$ with integer coefficients. I.e, there is a partition

$$[0,1]^n \; = \; X_0 \cup \cdots \cup X_m$$

and on each X_i

$$f(\vec{x}) \; = \; a_1 x_1 + \cdots + a_n x_n + a_0$$

for some $a_0, \ldots, a_n \in \mathbb{Z}$.

(blackboard)

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Theorem

$f : [0,1]^n \longrightarrow [0,1]^n$ is a product of McNaughton functions iff there are formulas $A_1, \ldots, A_n$ with $f = ([\![A_1]\!], \ldots, [\![A_n]\!])$.

Quasi-McNaughton functions

Rational numbers are definable by fixpoints:

$$\begin{aligned} \alpha \leftrightarrow \neg\alpha \quad &\Longrightarrow \quad \alpha = 1/2 \\ \alpha \leftrightarrow \neg(\alpha \cdot \alpha) \quad &\Longrightarrow \quad \alpha = 2/3 \end{aligned}$$

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Given a (product of) McNaughton function $g : [0,1]^{n+m} \longrightarrow [0,1]^n$ and $q_1, \ldots, q_m \in [0,1] \cap \mathbb{Q}$,

$$f(\vec{x}) := g(\vec{x}, \vec{q}) : [0,1]^n \longrightarrow [0,1]^n$$

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Lemma

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Proof. Given a quasi-McNaughton f, there are $A_1, \ldots, A_n$ and $q_1, \ldots, q_m \in [0,1] \cap \mathbb{Q}$ such that

$$f(\vec{x}) = ([\![A_1]\!](\vec{x}, \vec{q}), \ldots, [\![A_n]\!](\vec{x}, \vec{q})).$$

Con(Ł$_{fix}$) $\Rightarrow$ BFT

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The rationals $q_1, \ldots, q_m$ are definable by $\beta_i \leftrightarrow B_i(\beta_i)$ for $i = 1, \ldots, m$. Consider fixpoint equations for $\vec{A}(\vec{\alpha}, \vec{\beta}), \vec{B}(\vec{\beta})$.

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$$(r_1, \ldots, r_n, q_1, \ldots, q_m) \in [0,1]^{n+m}.$$

satisfying $\alpha_i \leftrightarrow A_i(\vec{\alpha}, \vec{\beta})$, that is, $\vec{r} = f(\vec{r})$.

Theorem

Con(Ł$_{fix}$) implies Brouwer's fixed point theorem.

Proof. Every continuous $f : [0,1]^n \longrightarrow [0,1]^n$ can be approximated by a sequence of quasi-McNaughton $\{f_i\}_{i \in \mathbb{N}}$:

$$f_i(x) \quad \to \quad f(x) \qquad (i \to \infty).$$

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We conclude $f(r) = r$.

A proof system for Ł$_{fix}$

(axiom Ł) is equivalent to

$$\frac{\Gamma, A \to B \Rightarrow B \quad \Delta, B \Rightarrow A \quad \Sigma, A \Rightarrow \Pi}{\Gamma, \Delta, \Sigma \Rightarrow \Pi} \text{ (Ł)}$$

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(cut) and (Ł) can be eliminated stepwise from derivations of $\Rightarrow$:

$$\frac{\overset{\vdots\, d_1}{A \to B \Rightarrow B} \quad \overset{\vdots\, d_2}{B \Rightarrow A} \quad \overset{\vdots\, d_3}{A \Rightarrow}}{\Rightarrow}\ (\text{Ł})$$

reduces to

$$\frac{\dfrac{\dfrac{\overset{\vdots\, d_3}{A \Rightarrow}}{A \Rightarrow B}}{\Rightarrow A \to B} \quad \overset{\vdots\, d_1}{A \to B \Rightarrow B} \quad \overset{\vdots\, d_2}{B \Rightarrow A} \quad \overset{\vdots\, d_3}{A \Rightarrow}}{\Rightarrow}\ (cut)$$

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Problem 3

Does the procedure terminate? If so, we would obtain a proof-theoretic proof of Brouwer's fixpoint theorem.

Systems to be discussed

	Fixpoints	Naive set theory
Łukasiewicz logic	$Ł_{fix}$	$Ł_{set}$
Monoidal t-norm logic	$\mathbf{MTL}_{fix}$	$\mathbf{MTL}_{set}$
Int. logic without contraction	$\mathbf{FLew}_{fix}$	$\mathbf{FLew}_{set}$

- Consistency of $Ł_{set}$ is a big open problem.

Terms and formulas:

$$
\begin{aligned}
t &::= x \mid \{x : \varphi(x)\} \\
\varphi &::= t \in t \mid 0 \mid \varphi \to \varphi \mid \forall x.\varphi
\end{aligned}
$$

$Ł_{set}$: extension of Ł∀ with naive comprehension axiom:

$$t \in \{x : \varphi(x)\} \quad \leftrightarrow \quad \varphi(t).$$

Łukasiewicz interpretation can be extended:

$$[\![\forall x.\varphi(x)]\!] \quad := \quad \bigwedge_{a \in D} [\![\varphi(a)]\!].$$

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Problem 4

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Is $Ł_{set}$ consistent?

Two obstacles:

- Infinitary $\bigwedge$ breaks continuity.

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BFT no more available. Forced to work proof-theoretically.

Systems to be discussed

	Fixpoints	Naive set theory
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- $\mathbf{MTL}_{fix}$, $\mathbf{MTL}_{set}$ are more tractable.

$\mathbf{MTL} := \mathbf{FLew}$ with prelinearity:

$$(pl) \quad A \to B \vee B \to A.$$

$\mathbf{MTL}_{fix}$: given n formulas $A_1(\vec{p}), \ldots, A_n(\vec{p})$, there are constants $\alpha_1, \ldots, \alpha_n$ such that

$$\begin{array}{rcl} \alpha_1 & = & A_1(\alpha_1, \ldots, \alpha_n) \\ \vdots & & \vdots \\ \alpha_n & = & A_n(\alpha_1, \ldots, \alpha_n) \end{array}$$

Formulas are identified modulo the equivalence.

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Formulas are identified modulo the equivalence.
$\mathbf{MTL}$ is a sublogic of Ł, so:

Fact

$\mathbf{MTL}_{fix}$ is consistent.

Hypersequent calculus for MTL_{fix}

Hypersequents: $\Theta_1 \mid \cdots \mid \Theta_n$ with Θ_i a sequent.
Hypersequent calculus for $\mathbf{FL}$ consists of

Rules of $\mathbf{FL}$

$$\frac{\Xi \mid A, \Gamma \Rightarrow B}{\Xi \mid \Gamma \Rightarrow A \rightarrow B}$$

Ext-Contraction

$$\frac{\Xi \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{\Xi \mid \Gamma \Rightarrow \Pi}$$

Communication

$$\frac{\Xi \mid \Gamma_1, \Delta_1 \Rightarrow \Pi \quad \Xi \mid \Gamma_2, \Delta_2 \Rightarrow \Lambda}{\Xi \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi \mid \Delta_1, \Delta_2 \Rightarrow \Lambda} \ (com)$$

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Rules of $\mathbf{FL}$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \quad \beta \Rightarrow \beta}{\alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha}\ (com)}{\Rightarrow \alpha \to \beta \mid \ \Rightarrow \beta \to \alpha}\ (\to r)}{\Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha) \mid \ \Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\ (\vee r)}{\Rightarrow (\alpha \to \beta) \vee (\beta \to \alpha)}\ (EC)$$

Goal: define a notion of size and design a shrinking cut elimination procedure.

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A slice of derivation d is a selection of 0 or 1 sequent from each hypersequent in d such that:

$$\frac{\Xi \mid A, \Gamma \Rightarrow B}{\Xi \mid \Gamma \Rightarrow A \to B} \qquad \frac{\Xi \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{\Xi \mid \Gamma \Rightarrow \Pi} \quad \text{or} \quad \frac{\Xi \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{\Xi \mid \Gamma \Rightarrow \Pi}$$

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The size $|d|$ is a multiset of natural numbers defined by:

$$|d| \quad := \quad \{|d'| : d' \text{ is a slice of } d\}.$$

where $|d'|$ is the number of inference rules visible in d'. We consider multiset ordering (which is well founded).

There is a shrinking cut elimination procedure for derivations of contradiction $\Rightarrow$.

$$\dfrac{\begin{array}{c}\vdots\, d_A\\ \Rightarrow A\end{array} \qquad \begin{array}{c}\vdots\, d_B\\ \Rightarrow B\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\, d_{AA}\\ A, A \Rightarrow\end{array} \qquad \begin{array}{c}\vdots\, d_{BB}\\ B, B \Rightarrow\end{array}}{A, B \Rightarrow \mid A, B \Rightarrow}\,(com)}{A, B \Rightarrow}\,(EC)}{\Rightarrow}\,(cut)$$

reduces to

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Theorem

Any proof of $\Rightarrow$ can be reduced to a cut-free proof (which does not exist).

Consistency of $\mathbf{MTL}_{set}$

The previous argument works for $\mathbf{MTL}_{set}$ as well.

Theorem

$\mathbf{MTL}_{set}$ is consistent.

Summary

	Fixpoints	Naive set theory
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- Hence by proving the former proof-theoretically, we obtain a new proof of the latter.

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- Consistency of Ł_{fix} is equivalent to Brouwer's fixpoint theorem.
- Hence by proving the former proof-theoretically, we obtain a new proof of the latter.
- Moreover, such a proof most likely extends to naive set theory, which would lead to the consistency of Ł_{set}, a big open problem in fuzzy mathematics.

Traditional logical approach to computational complexity

Formula-dependent characterizations of complexity classes.

polynomial hierarchy, bounded arithmetic, circuit/proof complexity, finite model theory, (safe recursion),

Eg. complexity of normalization in simply-typed λ-calculus:

Order	Complexity
2	P
3	PSPACE
4	$\mathrm{EXPTIME}$
5	$\mathrm{EXPSPACE}$
$\vdots$	$\vdots$
$2r+2$	$r\text{-}\mathrm{EXPTIME}$
$2r+3$	$r\text{-}\mathrm{EXPSPACE}$

$$\begin{array}{lcl} o(p) & := & 0 \\ o(A \to B) & := & \max\{o(A)+1, o(B \end{array}$$

$(Terui2012$

Girard's approach to computational complexity

"Intrinsic" complexity of a logic should be formula-independent and proof-theoretic (cut elimination).

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First check: consistency with fixpoints

adding fixpoints	$\approx$	ignoring formula complexity
consistency	$\approx$	termination of cut elimination

$\mathbf{FLew}_{fix}$ is:

- ☐ inconsistent with S4 (linear logic), KT, KD4
- ☐ consistent w. K (elementary logic), K-bounded (light logic)

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Connection with Brouwer-Łukasiewicz?

consistency with fixpoints $\approx$ continuous interpretation of formula