Instability, contraction, and truth

Shawn Standefer
University of Melbourne

December 11, 2015
The truth rules would be nice

\[
\frac{X, A \vdash Y}{X, T(\neg A)} \quad \frac{X \vdash A, Y}{X \vdash T(\neg A), Y}
\]

\((T\vdash)\) \quad \((\vdash T)\)

But they lead to paradoxes with minimal assumptions.
$L$ is equivalent to $\sim T(\lceil L \rceil)$.

\[
\begin{align*}
L & \vdash L & (T\vdash) \\
T(\lceil L \rceil) & \vdash L & (\vdash L) \\
\vdash L, \sim T(\lceil L \rceil) & & (\vdash \sim) \\
\vdash L, L & & (\vdash W) \\
\vdash L & & (\sim \vdash)
\end{align*}
\]

\[
\begin{align*}
L & \vdash L & (\vdash L) \\
L & \vdash T(\lceil L \rceil) & (\vdash T) \\
L, \sim T(\lceil L \rceil) & \vdash & (\sim \vdash) \\
L, L & & (L\vdash) \\
L & \vdash & (W\vdash) \\
L & & (\text{cut})
\end{align*}
\]

$\vdash$
Non-contractive theories of truth deal with the semantic paradoxes is to reject the structural rule of contraction.

\[
\frac{X, A, A, Z \vdash Y}{X, A, Z \vdash Y} \quad (W\vdash) \quad \frac{X \vdash Y, A, A, Z}{X \vdash Y, A, Z} \quad (\vdash W)
\]

Focus is on structural rule of contraction, rather than conditional form.
One question for proponents of non-contractive theorists is why contraction fails.

There’s been a lot of interesting work on this question recently, e.g. Mares and Paoli (2014), Beall and Murzi (2013), Shapiro (2015), etc.

I’m going to focus on one sort of answer, presented by Elia Zardini.
Zardini’s suggestion is that sentences express states of affairs, and some of these states of affairs are unstable.

*I think that the key to understanding what it is about the state-of-affairs expressed by a sentence that explains its failure to contract is given by thinking of that state-of-affairs as distinctively unstable. I conceive of instability as the metaphysical property attaching to states-of-affairs exemplification of which causes the exemplification of the logical property of failing to contract attaching to the corresponding sentences.*

(Zardini (2011, 504), emphasis in the original.)

Some states of affairs are stable and can obtain with all of their consequences, while some cannot obtain with their consequences.
The liar sentence, $L$, provides the paradigm example.

If the state of affairs represented by $L$ obtained, then so would that of $\sim L$, but these two states are incompatible.

Thus, the behavior of an unstable state-of-affairs as the one expressed by $[\sim L]$ resembles the behavior of physical states: both kinds of states, if they obtained, would lead to other states with which they would not co-obtain—although it is precisely the obtaining of the former states that would lead to the obtaining of the latter states. (Zardini (2011, 504))
As Zardini remarks, it is not entirely clear what states of affairs paradoxical sentences represent.

Much clearer in the case of “Snow is white”.

That is not the issue on which I will focus.
The temporal dimension is of course absent in the case of an unstable state-of-affairs as the one expressed by \( \sim L \), but there too we can discern a broadly analogous structure of ‘stages of truth evaluation,’ with the transition from one stage to the other being governed by \( T\)-introduction (for evaluations of truth) and (the contrapositive of) \( T\)-elimination (for evaluations of untruth). (Zardini (2011, 505))

Zardini says, in a subsequent footnote, that he is “appropriating and pushing into a certain direction some themes of the revision-theoretic tradition.” (Zardini (2011, 505, fn. 13))
Zardini, and all non-contractive theorists, defend naive truth, i.e. that truth obeys the rules:

\[
\frac{X, A \vdash Y}{X, T(\neg A\neg) \vdash Y} \quad \text{\texttt{(T)}}
\]

\[
\frac{X \vdash A, Y}{X \vdash T(\neg A\neg), Y} \quad \text{\texttt{(+T)}}
\]

The models for these theories are fixed-point models, first introduced by Kripke (1975).
Simple fixed-point picture

One begins with a three- (or more) valued interpretation for truth, $f_0$, which assigns all sentences a minimal value in an ordering on the semantic values, e.g. $\frac{1}{2}$.

Using the model for the base language, together with $f_0$, sentences are evaluated, and the ones that receive value 1 are added to the extension of the truth predicate (and similarly for 0 and the anti-extension) to get a new interpretation, $f_1$.

This process is monotonic, in the sense that once a sentence is in the (anti-)extension, then it does not leave.

This process is continued transfinitely, $f_0, f_1, \ldots, f_\omega, \ldots$, and it eventually reaches a fixed-point, $f_\alpha = f_{\alpha+1}$, which is the final interpretation of the truth predicate.
The point to which I want to draw attention is that instability has no place in fixed-point models.

This suggests that alternative models would be a better way to understand Zardini’s states of affairs.

Following Zardini’s suggestion, I will use revision theory to do so.
Unlike fixed-point approaches, revision theory naturally accommodates instability.

It is, arguably, one of its key features.

Zardini’s quotation suggests that he thinks revision theory is a good model for understanding contraction failure.

Let’s turn to revision theory.
Revision theory of truth

Revision theory was developed as a theory of truth by Gupta and Herzberger.

The T-sentences, $T(\neg A \neg) \iff A$, tell us how to revise the extension of truth.

There is a sequence of evaluations, and when $A$ is 1 at one stage, $T(\neg A \neg)$ is 1 at the next.

This revision is carried out in the classical scheme, so sentences like the liar can jump in and out of the truth predicate’s extension.
Revision theory: formalism

Base language $\mathcal{L}$ interpreted via a classical ground model $M(=\langle D, I\rangle)$.

Expand language to $\mathcal{L}^+$ with $T$ and names for all sentences, if not already available.

A hypothesis $h$ is a function from sentences to classical truth values – $h : \text{Sents}_{\mathcal{L}^+} \mapsto \{1, 0\}$.

Hypotheses interpret truth.
Semantics

Model $M + h$ is just like $M$ except that $h$ interprets the truth predicate

$$Val_{M+h}(T(\neg A)) = h(A),$$
and set non-sentences to 0.
The set of T-sentences, $T(\neg \neg A) =_{Df} A$ yields a revision operator $\tau_M$ for each model $M$.

Obeys the equivalence

$$\tau_M(h)(T(\neg \neg A)) = Val_{M+h}(A).$$
Sequences

\[ h, \tau_M(h), \tau_M(\tau_M(h)), \tau_M^3(h), \ldots \]

Sequences of applications of \( \tau_M \) to a hypothesis will define validity.

Interested in just eventual stability in \( \omega \)-sequences.
To aid the comparison with Zardini’s theory, I will give a sequent system for the revision theory of truth.

The system uses indexed formulas, $A^i$.

Here, the indices can be any natural number.

The system uses multisets on either side of the turnstile.
Calculus: connectives

\[ A^i \vdash A^i \]

\[ X, A^i \vdash Y \quad \frac{X \vdash \sim A^i, Y}{X \vdash \sim A^i, Y} \quad (\vdash \sim) \]

\[ X \vdash A^i, Y \quad \frac{X \vdash A^i, Y}{X, \sim A^i \vdash Y} \quad (\sim \vdash) \]

\[ X \vdash A^i, Y \quad \frac{X \vdash A^i, Y}{X \vdash B^i, Y} \quad (\vdash A \& B) \]

\[ X \vdash A^i, Y \quad \frac{X \vdash B^i, Y}{X, A \& B^i \vdash Y} \quad (\vdash \& \vdash) \]

\[ X, A \& B^i \vdash Y \]

\[ X, B^i \vdash Y \quad (\& \vdash) \]
Calculus: truth

\[
\frac{X, A^i \vdash Y}{X, T(\neg A)^i \vdash Y} \quad (T\vdash) \\
\frac{X \vdash A^i, Y}{X \vdash T(\neg A)^i \vdash, Y} \quad (\vdash T)
\]
**Calculus: structural rules**

\[ \frac{X, A^i, A^i \vdash Y}{X, A^i \vdash Y} \quad (W\vdash) \]

\[ \frac{X \vdash Y, A^i, A^i}{X \vdash Y, A^i} \quad (\vdash W) \]

\[ \frac{X \vdash Y}{X, A^i \vdash Y} \quad (K\vdash) \]

\[ \frac{X \vdash Y}{X, A^i \vdash Y} \quad (\vdash K) \]

\[ \frac{X \vdash B^i, Y}{X \vdash B^k, Y} \quad (\vdash IS) \]

\[ \frac{X, B^i \vdash Y}{X, B^k \vdash Y} \quad (IS\vdash) \]
Interpreting

A sequent $A_{1}^{k_{1}}, \ldots, A_{i}^{k_{i}} \vdash B_{1}^{m_{1}}, \ldots, B_{n}^{m_{n}}$ is satisfied by $M + h$ just in case if for each $A_{i}^{k_{i}}$, $Val_{M + \tau^{k_{i}}(h)}(A_{i}) = 1$, then some $B_{n}^{m_{n}}$ has $Val_{M + \tau^{m_{n}}(h)}(B_{n}) = 1$.

A sequent is valid in $M$ just in case there is some natural number $n$ such that for all $m \geq n$, it is satisfied in $M + \tau^{m}(h)$.

A sequent is valid just in case it is valid in $M$ for all classical ground models $M$.

This is sound for validity, and there is a slightly adjusted system that is complete.
Example

Table: Sample revision for the liar and a base language sentence

<table>
<thead>
<tr>
<th></th>
<th>stage 0</th>
<th>stage 1</th>
<th>stage 2</th>
<th>stage 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$T(\neg L)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$Fa$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$T(\neg Fa)$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
How does contraction fail?

The truth rules increment the indices, but contraction requires the same index on both copies of the formula.
Contraction

\[
\begin{align*}
\frac{L^i \vdash L^i}{L^i \vdash T(\neg L^\bot)^{i+1}} \quad (\vdash T) \\
\frac{L^i \vdash T(\neg L^\bot)^{i+1}}{L^i, \sim T(\neg L^\bot)^{i+1} \vdash} \quad (\sim \vdash) \\
\frac{L^i, \sim T(\neg L^\bot)^{i+1} \vdash}{L^i, L^{i+1} \vdash} \quad (L^\bot) \\
\frac{L^i, L^{i+1} \vdash}{L^i \vdash} \quad (??)
\end{align*}
\]

Can’t contract \(L^i\) and \(L^{i+1}\)

The clash of indices will prevent contraction on other paradoxical sentences as well.
Stability

For sentences that are stable, one can contract freely.

For example, the hereditarily T-free sentences permit free contraction.

Truth applied to sentences that are classical tautologies are also fine to contract.

It’s really just the unstable sentences that lead to problems with contraction.
While revision theory motivates the contraction failures that Zardini wants, it cannot be the whole story.

The reason is that it equally motivates rejecting many things that he wants to maintain.
Problem: truth

Zardini’s preferred truth rules are the naive ones, according to which all iterations of $T$ are equivalent.

This is not sustained by revision theory.

The liar provides an example of this equivalence failing, since $L$ and $T(\neg L \downarrow)$ are bound to differ after revision.
Problem: connectives

The issue with truth spreads to the other connectives.

To apply ($\vdash \&$), both conjuncts must have the same index.

Non-contractive approaches do not have any such requirements.
Problem: additive connectives

\[ X \vdash A, Y \quad X \vdash B, Y \]
\[ \frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \& B, Y} \quad (\vdash \&) \]

\[ X, A \vdash Y \]
\[ \frac{X, A \vdash Y}{X, A \& B \vdash Y} \quad (& \vdash) \]

\[ X, B \vdash Y \]
\[ \frac{X, B \vdash Y}{X, A \& B \vdash Y} \quad (& \vdash) \]
Problem: additive connectives

\[
\begin{align*}
A & \vdash A \\
A & \vdash T(\neg A) \\
\hline
A & \vdash A \land T(\neg A) & (\vdash \land) \\
\end{align*}
\]

This will generally be invalid in revision theory.

The indices won’t line up properly.
Problem: multiplicative connectives

\[
\begin{align*}
X &\vdash A, B, Y \\
X &\vdash A \oplus B, Y \\
X, A &\vdash Y \\
U, B &\vdash V \\
X, U, A \oplus B &\vdash Y, V
\end{align*}
\]
Problem: multiplicative connectives

\[
\begin{align*}
T(\Gamma A) & \vdash A \quad (T\vdash) \\
\vdash A, \sim T(\Gamma A) & \quad (\vdash\sim) \\
\vdash A \oplus \sim T(\Gamma A) & \quad (\vdash\oplus)
\end{align*}
\]

This will generally be invalid in revision theory, since the two ‘disjuncts’ will generally have different indices.

Natural ways of adding index-ignoring multiplicatives to revision theory result in triviality or loss of commutativity.
The use of revision theory supports Zardini’s idea about contraction.

It equally well supports rejecting many other principles governing truth and connectives (and quantifiers) that non-contractive theorists wish to maintain.

The most straightforward ways of understanding the idea of instability leads to these failures.

The non-contractive theorist needs to supply some additional reasons for focusing on just the contraction failure of revision theory or to supply a different story for the failure of contraction.
Thank you

Thanks!

Questions? Comments?


