A fresh look at the Inozemtsev spin chain

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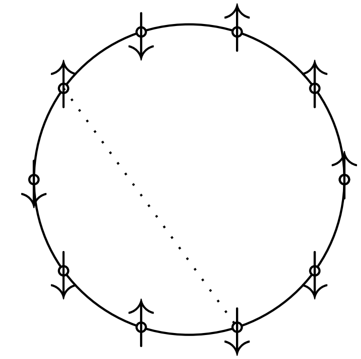
ongoing work with **Rob Klabbers** (NORDITA)

building on [V. I. Inozemtsev, '90–'99]

Exactly solvable long-range spin chains



	nearest-neighbour (contact)	intermediate (elliptic)	inverse-square (trigonometric)
anisotropic (elliptic)	(some version of) Heisenberg XYZ	anisotropic Inozemtsev ??	anisotropic Haldane–Shastry ??
partially isotropic (trig)	(some version of) Heisenberg XXZ	*q*-deformed Inozemtsev ?	*q*-deformed Haldane–Shastry [Didina's talk]
isotropic (rational)	Heisenberg XXX	Inozemtsev [this talk]	Haldane–Shastry

Anatomy of Inozemtsev's elliptic spin chain

[Inozemtsev '90, '95]

Weierstrass $\wp$, periods $(L,\omega) \in \mathbb{N} \times \mathrm{i}\,\mathbb{R}_{>0}$

long-range pairwise → elliptic potential ← spin exchange

$$H = \sum_{i<j}^{L} \frac{\sin^2(\pi/\omega)}{(\pi/\omega)^2} \left(\wp(i-j) + \frac{2}{\omega}\,\zeta\!\left(\frac{\omega}{2}\right) \right) \frac{1 - \vec{\sigma}_i \cdot \vec{\sigma}_j}{4}$$

$\omega \to \mathrm{i}0$: $\delta_{d(i-j),1}$ [Heisenberg '28]

$L \to \infty$:

$$\frac{\sinh^2 \kappa}{\sinh^2 \kappa(i-j)} = \frac{\sin^2(\pi/\omega)}{\sin^2(\pi(i-j)/\omega)}$$

$\frac{\mathrm{i}\,\pi}{\kappa} = \omega$

[Inozemtsev '92]

$\omega \to \mathrm{i}\infty$: $\frac{(\pi/L)^2}{\sin^2(\pi(i-j)/L)}$ [Haldane '88] [Shastry '88]

Some highlights
Inozemtsev's elliptic spin chain

- Spectrum for M=2 [Inozemtsev '90]
  using Hermite's solution for **Lamé equation**
- Spectrum in **hyperbolic limit** [Inozemtsev '92]
  via connection to **hyperbolic Calogero–Sutherland** using [Chalykh Veselov '90]
- Spectrum for **general** M [Inozemtsev '95, '99]
  via connection to **elliptic Calogero–Sutherland** using [Felder Varchenko '95]
- Asymptotic **Yangian** symmetry [Ha Haldane '93] [De La Rosa Gomez *et al* '16]
- **Thermodynamic** Bethe ansatz [Dittrich Inozemtsev '97] [Klabbers '16]
- **Proposal** for **higher Hamiltonians** [Inozemtsev '96]
- Guest appearance in **AdS/CFT** [Serban Staudacher '04]

Our goal

$$H = \sum_{i<j}^{L} \frac{\sin^2(\pi/\omega)}{(\pi/\omega)^2} \left(\wp(i-j) + \frac{2}{\omega}\,\zeta\left(\frac{\omega}{2}\right) \right) \frac{1 - \vec{\sigma}_i \cdot \vec{\sigma}_j}{4}$$

Understand Inozemtsev's solution

and its **limits**

Strategy

General considerations

- **Isotropy** ($\mathfrak{sl}_2$-invariance): fix $M = \# \downarrow$, focus on h.w.
- Use coordinate basis $\sigma^-_{n_1} \cdots \sigma^-_{n_M} |\uparrow \cdots \uparrow\rangle$
  (so will have to check **cyclicity**)
- Homogeneity (translational invariance) determines $M = 1$
  Dispersion relation

[Inozemtsev '90]
[Klabbers JL]

$$\omega = \frac{\mathrm{i}\,\pi}{\kappa}$$

$$\varepsilon(p) = \sinh^2\!\kappa \left(\bar{\wp}(p) - \left(\bar{\zeta}(p) - \frac{\bar{\zeta}(\pi)}{\pi}\, p \right)^2 - \frac{\bar{\zeta}(\pi)}{\pi} \right)$$

$\kappa \to \infty$: $2 \sin^2(p/2)$ — Heisenberg

reciprocal periods $(2\pi, 2\mathrm{i}\kappa)$

$\kappa \to 0$: $\frac{1}{4}\, p\,(2\pi - p)$ — Haldane–Shastry

Strategy
Extended coordinate Bethe ansatz

[Inozemtsev '95]
[Klabbers JL]

For general M seek wave functions of the form

$$\Psi_{\boldsymbol{p}}(\boldsymbol{n}) = \sum_{w \in S_M} \widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}_w)\, \mathrm{e}^{\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{n}_w}$$

$\boldsymbol{p} := \tilde{\boldsymbol{p}} + \boldsymbol{r}$

depends on **positions**!

plane wave with temporary parameter

Assumptions

- [technical] $\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}$ has **simple poles** at **equal arguments**
- **double quasiperiodicity** $\begin{cases} \widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}+\omega\hat{e}_m) = \mathrm{e}^{\mathrm{i}\omega\tilde{p}_m}\,\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}) \\ \widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}+L\hat{e}_m) = \mathrm{e}^{\mathrm{i}(L\tilde{p}_m-\varphi_m)}\,\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}) \end{cases}$
- **Bethe-ansatz equations** (implies cyclicity of $\Psi_{\boldsymbol{p}}$)

$$L\, p_m = 2\pi I_m + \varphi_m\,, \qquad I_m \in \mathbb{Z}_L\,, \qquad 1 \leq m \leq M$$

Connection with elliptic Calogero–Sutherland

If $\widetilde{H}^{(2)}\,\widetilde{\Psi}_{\tilde{\boldsymbol{p}}} = \widetilde{E}\,\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}$ for **quantum ellCS**

$$\widetilde{H}^{(g)} = -\frac{1}{2}\sum_{m=1}^{M}\partial_{x_m}^2 + g(g-1)\sum_{m<m'}^{M}\wp(x_m - x_{m'})$$

(with simple poles at equal arguments)

and if **BAE** $\;L\,p_m = 2\pi I_m + \varphi_m\,,\;\; I_m \in \mathbb{Z}_L\,,\;\; 1 \leq m \leq M$

then '**freezing**' $x_m = n_m \in \mathbb{Z}_L\,,\; \tilde{p}_m = \lambda(p_m)$ [→ next slide]

gives **solution** $\Psi_{\boldsymbol{p}}(\boldsymbol{n}) = \sum_{w\in S_M}\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{n}_w)\,\mathrm{e}^{\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{n}_w},\quad \boldsymbol{r} = \boldsymbol{p} - \tilde{\boldsymbol{p}}$

with

$$p = \sum_{m=1}^{M}\Big(p_m - \frac{\varphi_m}{L}\Big) \bmod 2\pi\,,\qquad E = \sum_{m=1}^{M}\varepsilon(p_m) + \widetilde{U}$$

↗ $\widetilde{E} = \frac{1}{2}\|\tilde{\boldsymbol{p}}\|^2 + \widetilde{U}$

Comments

- If $\sum \varphi_m = 0$ then $p = \sum p_m \bmod 2\pi$ **quasimomenta**
- $\tilde{p}_m = \lambda(p_m)\,,\ \lambda(p) = -2\kappa\Big(\bar{\zeta}(p) - \frac{\bar{\zeta}(\pi)}{\pi}\,p\Big)$ **rapidities**

$$2\kappa \times -\tfrac{1}{2}\cot\tfrac{p}{2} \quad \xleftarrow{\ \kappa \to \infty\ } \qquad \xrightarrow{\ \kappa \to 0\ } \quad p - \pi$$

- $E = \sum \varepsilon(p_m) + \widetilde{U}$ is **additive** iff $\widetilde{E} = \frac{1}{2}\|\tilde{\boldsymbol{p}}\|^2 + \widetilde{U}$ is so (which doesn't seem to be the case)
- $\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}$ will have simple poles **or double zeroes** at equal arguments; latter would be better in HS limit
- Parameter $\boldsymbol{r} = \boldsymbol{p} - \tilde{\boldsymbol{p}}$ drops out

Two-magnon solution

- **Lamé** $\widetilde{H}^{(2)} = -\frac{1}{2}\big(\partial_{x_1}^2 + \partial_{x_1}^2\big) + 2\,\wp(x_1 - x_2)$

$$\widetilde{\Psi}_{\tilde{\boldsymbol{p}}}(\boldsymbol{x}) = \mathrm{e}^{2\zeta(\omega/2)\,(x_1-x_2)\,\gamma/\omega}\,\frac{\sigma(x_1-x_2-\gamma)}{\sigma(x_1-x_2)\,\sigma(\gamma)}\,\mathrm{e}^{\mathrm{i}\tilde{\boldsymbol{p}}\cdot\boldsymbol{x}}$$

$$\varphi_1 = -\varphi_2 = -\gamma,\;\; \widetilde{U} \sim -\varepsilon\big|_{\kappa\to L\,\kappa}(\gamma)$$

- **Heisenberg limit** get Bethe ansatz, and BAE: [Klabbers JL]

$$\left(\frac{\lambda_{\mathrm{H}}(p_1)-\mathrm{i}/2}{\lambda_{\mathrm{H}}(p_1)+\mathrm{i}/2}\right)^L = \frac{\lambda_{\mathrm{H}}(Lp_1)-\mathrm{i}/2}{\lambda_{\mathrm{H}}(Lp_1)+\mathrm{i}/2} = \frac{\lambda_{\mathrm{H}}(p_1)-\lambda_{\mathrm{H}}(p_2)-\mathrm{i}}{\lambda_{\mathrm{H}}(p_1)-\lambda_{\mathrm{H}}(p_2)+\mathrm{i}}$$

- In **elliptic coordinates** the **BAE become polynomial**; can count roots to check **completeness**; **Energy** becomes rational too [Klabbers JL]

General solutions of eCS

- **Bethe-type** expression [Felder Varchenko '95]
  - Explicit, has simple poles at equal arguments
  - $M(M-1)/2$ auxiliary parameters with their own 'Bethe-like equations'
- **Perturbative** expressions [Takemura '00] [Komori Takemura '02]
  - Double zeroes at equal arguments
  - 'Elliptic Jack polynomials' [Langmann '00 '14]
- Nonstationary Ruijsenaars functions [Shiraishi '19]

Conclusion

- Solution via **relation with $g = 2$ elliptic Calogero–Sutherland**
- **'Quasi-additive' energy $E = \sum_m \varepsilon(p_m) + \widetilde{U}$**
- We're starting to get a grip on **limiting cases** for $M = 2$
- **Spectral problem rationalises** completely (algebraic)
- **Open questions**
  - **Heisenberg** and **Haldane–Shastry limits** in general?
  - **Pole-free derivation** ?
  - **Separation of variables** ??
  - **XXZ-like** (q-)analogue ?
  - **XYZ-like** analogue ??