CHALMERS UNIVERSITY OF TECHNOLOGY | UNIVERSITY OF GOTHENBURG

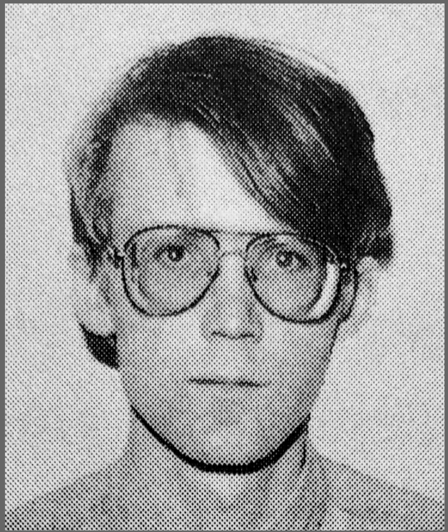
Resurrecting the *q*-deformed Haldane–Shastry model

Jules Lamers

arXiv:1801.05728

building on D. Uglov

arXiv:hep-th/950814



[Collected Papers of Denis Uglov]

Outline

- Brief recap of the Haldane–Shastry model
- The q-deformed Haldane–Shastry model

  definition   key properties   further properties
- Exact spectrum at finite length
- The fine structure

  relation with TL   Uglov's formula
- Conclusion

Brief recap of the Haldane–Shastry model

[Haldane '88]
[Shastry '88]

spin-1/2 chain, L sites

Hilbert space: $(\mathbb{C}^2)^{\otimes L}$

exchange interactions

$P_{ij} - 1 = \frac{1}{2}(\vec{\sigma}_i \cdot \vec{\sigma}_j - 1)$

$$H_{\mathrm{HS}} = -J \sum_{i<j}^{L} V_{\mathrm{HS}}(i-j)(P_{ij} - 1)$$

$J > 0$ ferromagnetic

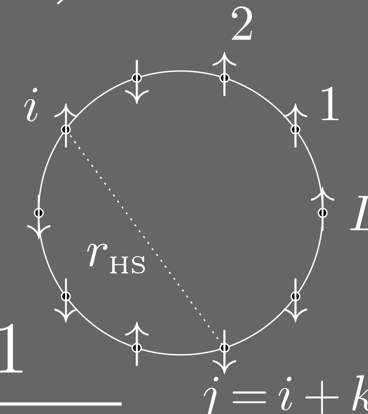
$J < 0$ antiferromagnetic

long-range pairwise interactions

pair potential

$$V_{\mathrm{HS}}(k) = \frac{1}{r_{\mathrm{HS}}(k)^2}$$

$$r_{\mathrm{HS}}(k) = 2\sin(\tfrac{\pi}{L}k)$$



Brief recap of the Haldane–Shastry model

- Remarkable **spectrum**:
  very **regular, highly degenerate** [Haldane '88]

- Origin:
  - **quadratic** dispersion
  - **additive** energies
  - **infinite-dimensional symmetry algebra** [Haldane *et al* '92]
    Yangian $Y(\mathfrak{sl}_2)$ generated by
    - 'level 0' $\vec{S} = \frac{1}{2}\sum_j \vec{\sigma}_j$ (**isotropy**)
    - 'level 1' $\vec{Q} = \frac{1}{2}\sum_{i<j} \cot\big[\frac{\pi}{L}(i-j)\big]\, \vec{\sigma}_i \times \vec{\sigma}_j$

Question

Heisenberg XXX : XXZ = HS : ?

- **Uglov** derived **Hamiltonian** and its **spectrum**
  - algebraic set-up of [Bernard *et al* ’93]
  - computational ingredient of [Talstra Haldane ’95]
- Today: **new, simpler expression** for the Hamiltonian

The *q*-deformed Haldane–Shastry model

[Uglov '95]
[JL '18]

spin-1/2 chain, L sites
Hilbert space: $(\mathbb{C}^2)^{\otimes L}$

q-deformed
exchange interactions

$$H = -J \sum_{i<j}^{L} V(i-j)\, S_{[ij]}$$

$J > 0$ ferromagnetic
$J < 0$ antiferromagnetic

long-range
pairwise form

point-split potential

$$V(k) = \frac{1}{r_+(k)\, r_-(k)}$$

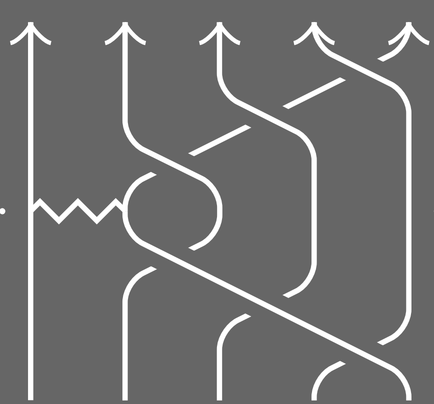
$$r_\pm(k) = 2\sin(\tfrac{\pi}{L}k \pm \mathrm{i}\gamma)$$

The *q*-deformed Haldane–Shastry model

Write $H = -J \sum_{i<j}^{L} V(i-j)\, S_{[i,j]}$ via $\begin{cases} z_j = \mathrm{e}^{2\pi\mathrm{i}\, j/L} \\ q = \mathrm{e}^{\gamma} \end{cases}$

so that $V = \dfrac{z_i\, z_j}{(q\, z_i - q^{-1} z_j)(q\, z_j - q^{-1} z_i)}$

The spin interactions are

$S_{[i,j]} = \cdots$  $\cdots$

$$= \check{R}\Big(\frac{v}{u}\Big) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(q-q^{-1})\, v}{q\, v - q^{-1} u} & \frac{v-u}{q\, v - q^{-1} u} & 0 \\ 0 & \frac{v-u}{q\, v - q^{-1} u} & \frac{(q-q^{-1})\, u}{q\, v - q^{-1} u} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (q - q^{-1})\, \check{R}'(1) = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The q-deformed Haldane–Shastry model
Key properties

[Uglov '95]

- Isotropic limit $q = \mathrm{e}^{\gamma} \to 1: \ H \to H_{\mathrm{HS}}$
- $q \neq \pm 1$ breaks $\mathfrak{su}_2 \rightsquigarrow \mathfrak{u}_1$: $[S^z, H] = 0$ (partial isotropy)
- Still an infinite-dimensional symmetry algebra:
  q deforms $Y(\mathfrak{sl}_2) \to U_q'(\widehat{\mathfrak{gl}_2})$ quantum-affine $\mathfrak{sl}_2$

  $\cup$ $\cup$

  'level 0' $\mathfrak{sl}_2 \to U_q(\mathfrak{sl}_2)$ $\begin{cases} [S^z, S_q^{\pm}] = \pm S_q^{\pm} \\ [S_q^+, S_q^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}} \end{cases}$

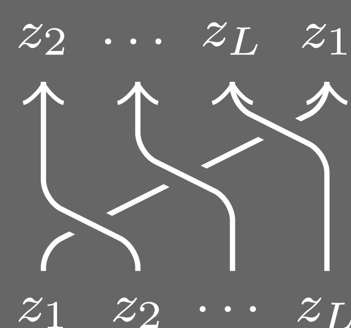
  + 'higher levels'
  (more complicated)

  $$S_q^{\pm} = \sum_j (q^{\sigma^z/2})^{\otimes(j-1)} \otimes \sigma^{\pm} \otimes (q^{-\sigma^z/2})^{\otimes(L-j)}$$
- Spectrum: level splitting, still additive

The q-deformed Haldane–Shastry model
Further properties

- **Hermitian** for $q = \mathrm{e}^{\gamma} \in \mathbb{R}$  $= -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ [JL ’18]

- **Multi-spin interactions**:
  transport by  in $S_{[i,j]}$ affects intermediate spins

- $q \neq \pm 1$ deforms translational invariance: $[U, H] = 0$
  with q-shift $U =$ 

  eigenvalues $\mathrm{e}^{\mathrm{i}\,p}$ (q-momentum); $\mathrm{e}^{\mathrm{i}\,p\,L} = 1$ [JL ’18]

- Invariant under simultaneous reversal $\begin{cases} |{\uparrow}\rangle \leftrightarrow |{\downarrow}\rangle \\ z_j \mapsto z_j^{-1} \\ q \mapsto q^{-1} \end{cases}$ [JL ’18]

Exact spectrum at finite length
Dispersion relation

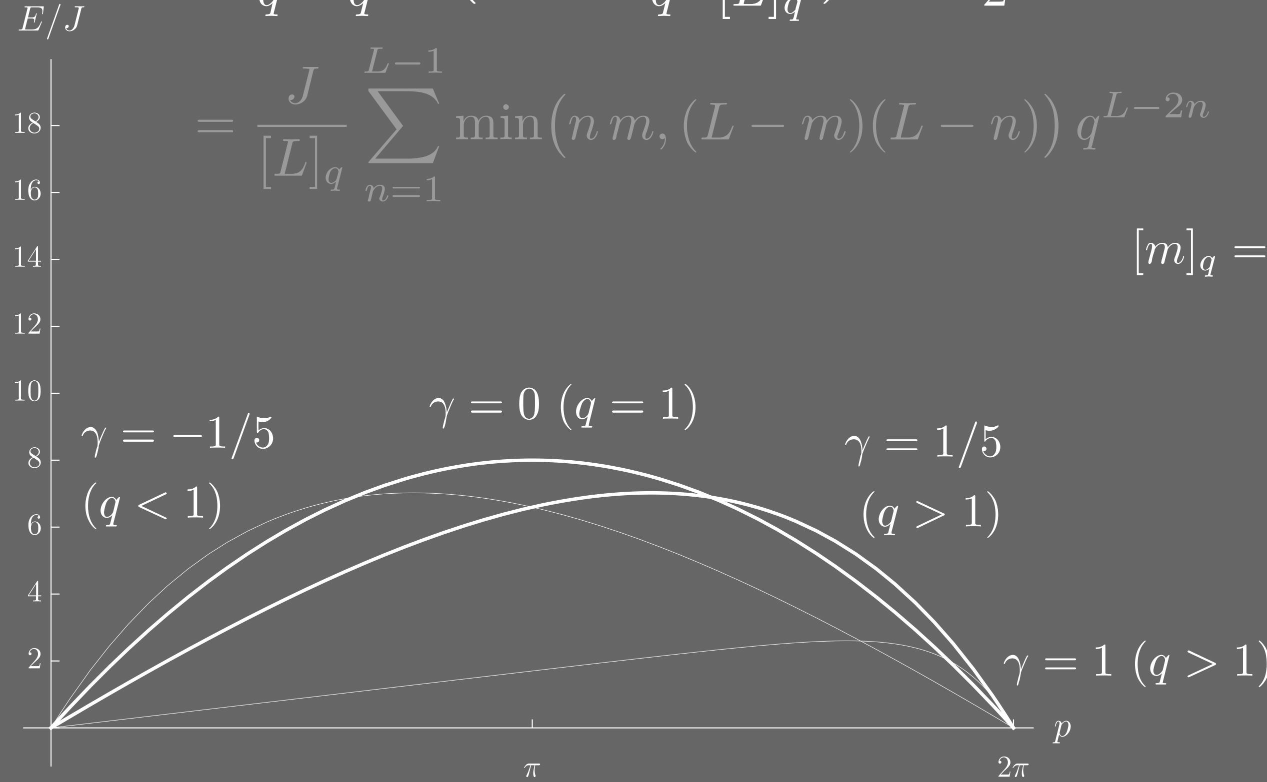
$$\varepsilon(m) = \frac{J}{q - q^{-1}} \Big(m - L\, \frac{q^m\, [m]_q}{q^L\, [L]_q} \Big) \;\to\; \frac{J}{2}\, m\,(L-m) = \varepsilon_{\text{HS}}(m)$$

[Uglov '95]

$$= \frac{J}{[L]_q} \sum_{n=1}^{L-1} \min\big(n\, m, (L-m)(L-n)\big)\, q^{L-2n}$$

[JL '18]

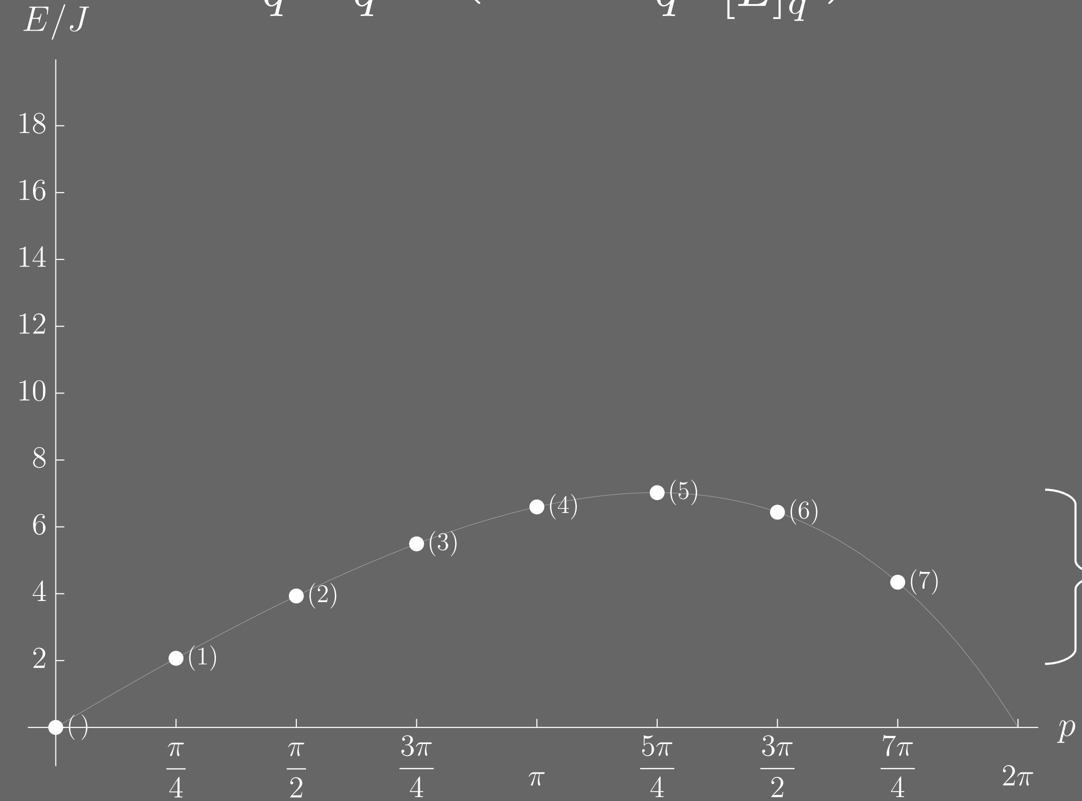
$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$$



$$p = \frac{2\pi\, m}{L}$$

Exact spectrum at finite length
Quantization of *q*-momentum

$\gamma = 1/5,\ L = 8$ $\qquad \varepsilon(m) = \dfrac{J}{q - q^{-1}} \left(m - L \, \dfrac{q^m \, [m]_q}{q^L \, [L]_q} \right)$ $\qquad [m]_q = \dfrac{q^m - q^{-m}}{q - q^{-1}}$



building blocks of spectrum

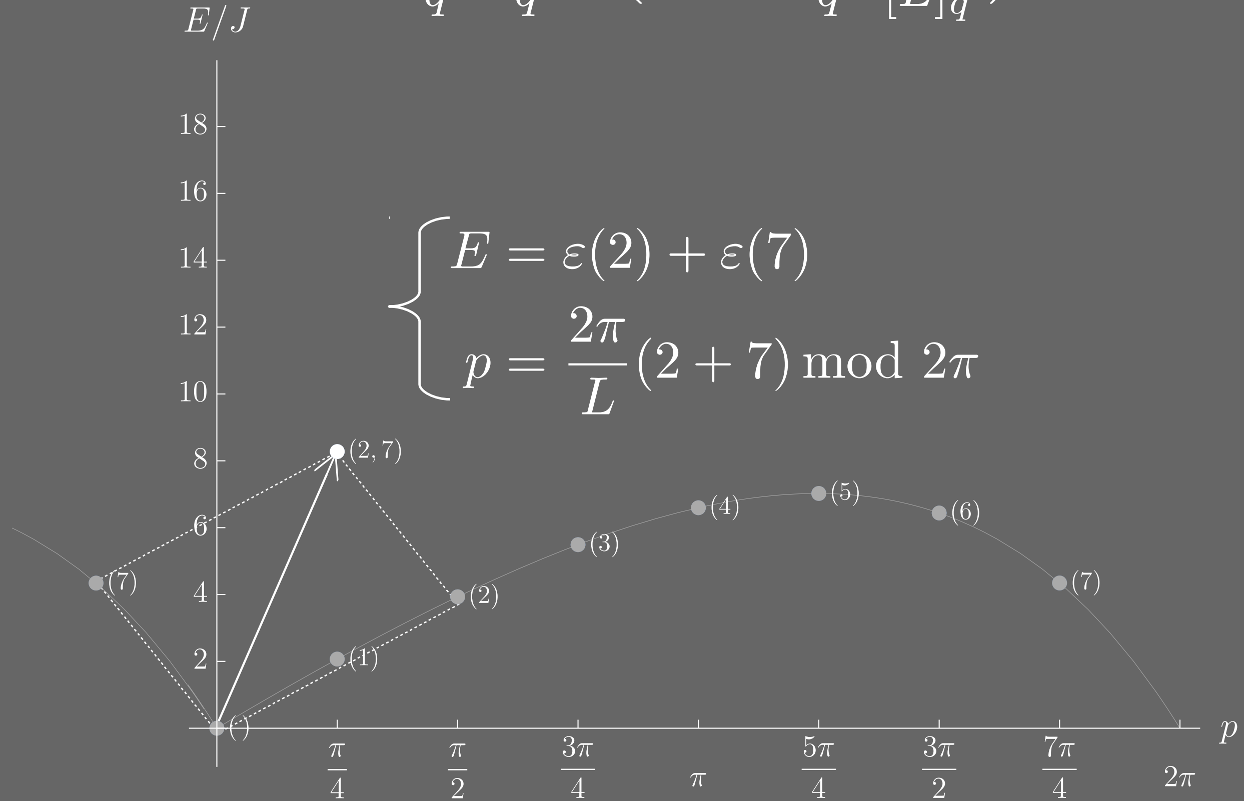
$$p = \frac{2\pi \, m}{L}$$

Exact spectrum at finite length
Additivity

$\gamma = 1/5, \ L = 8$

$$\varepsilon(m) = \frac{J}{q - q^{-1}} \left(m - L \, \frac{q^m \, [m]_q}{q^L \, [L]_q} \right)$$

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$$



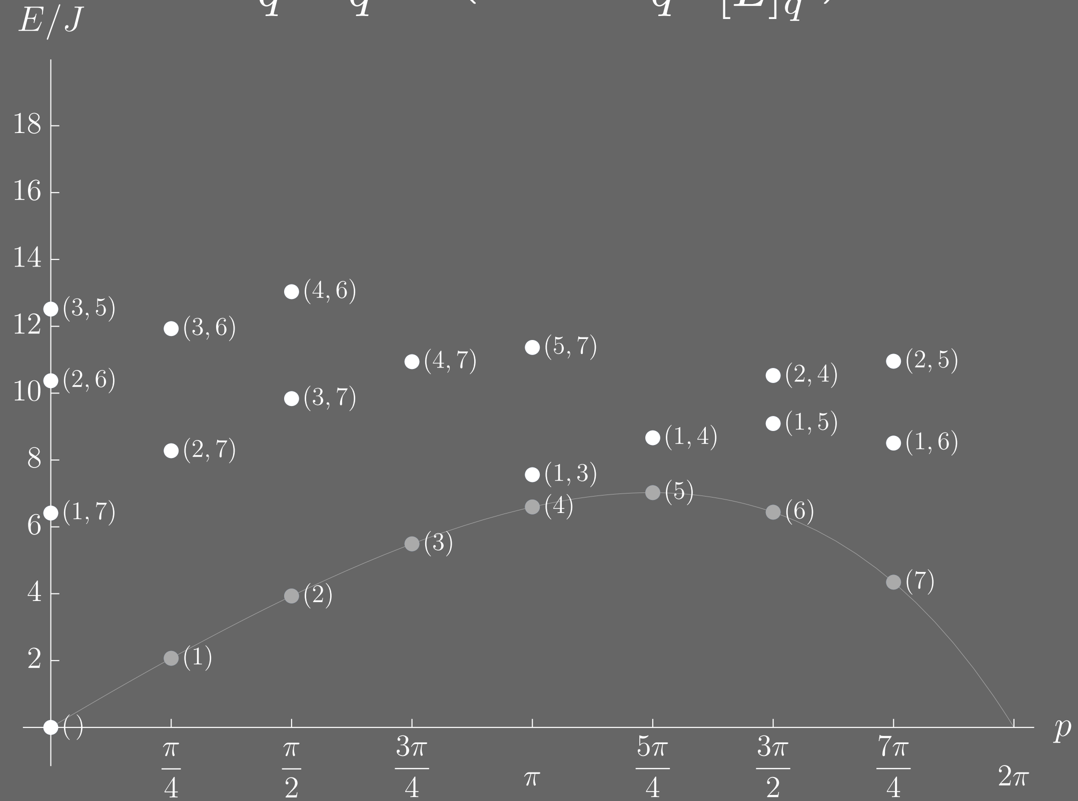
$$\begin{cases} E = \varepsilon(2) + \varepsilon(7) \\ p = \dfrac{2\pi}{L}(2+7) \bmod 2\pi \end{cases}$$

Exact spectrum at finite length
Motifs

$\gamma = 1/5, \; L = 8$

$$\varepsilon(m) = \frac{J}{q - q^{-1}} \left(m - L \, \frac{q^m \, [m]_q}{q^L \, [L]_q} \right) \qquad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$$

$$E = \varepsilon(m_1) + \varepsilon(m_2)$$



addition is allowed iff $m_2 > m_1 + 1$

$$p = \frac{2\pi}{L}(m_1 + m_2) \bmod 2\pi$$

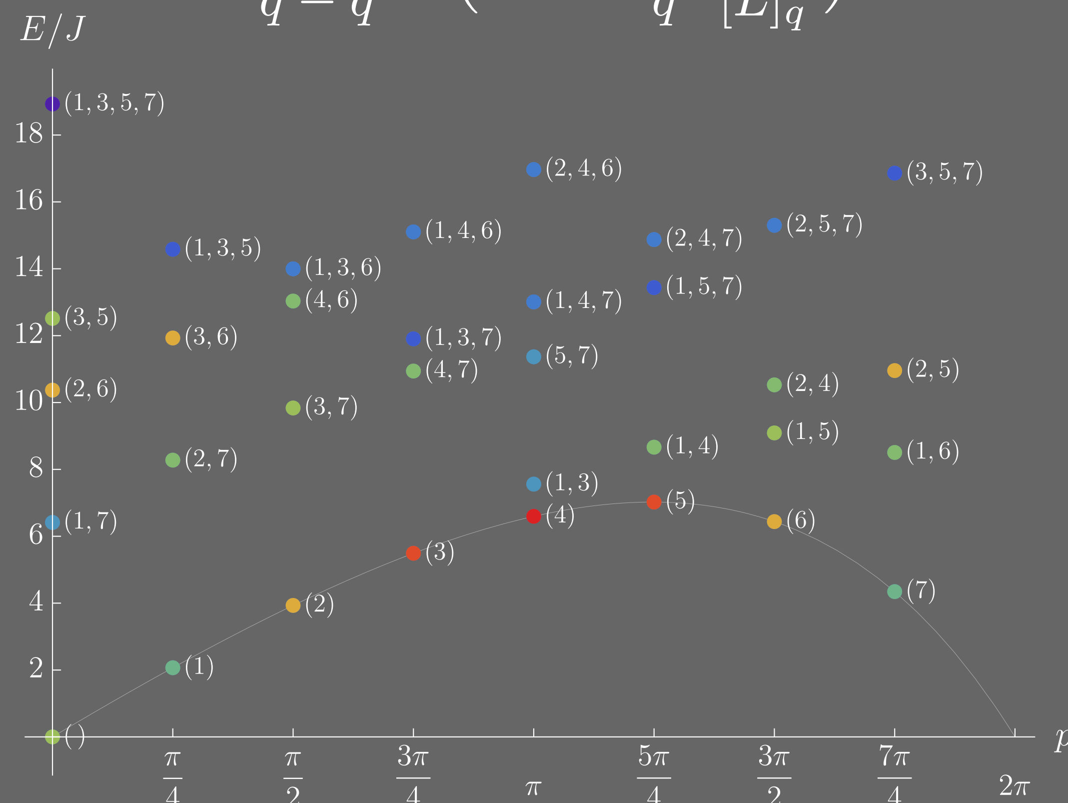
Exact spectrum at finite length

Complete spectrum

$\gamma = 1/5, \; L = 8$

$$\varepsilon(m) = \frac{J}{q - q^{-1}} \left(m - L \, \frac{q^m \, [m]_q}{q^L \, [L]_q} \right) \qquad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$$

$$E = \sum_r \varepsilon(m_r)$$

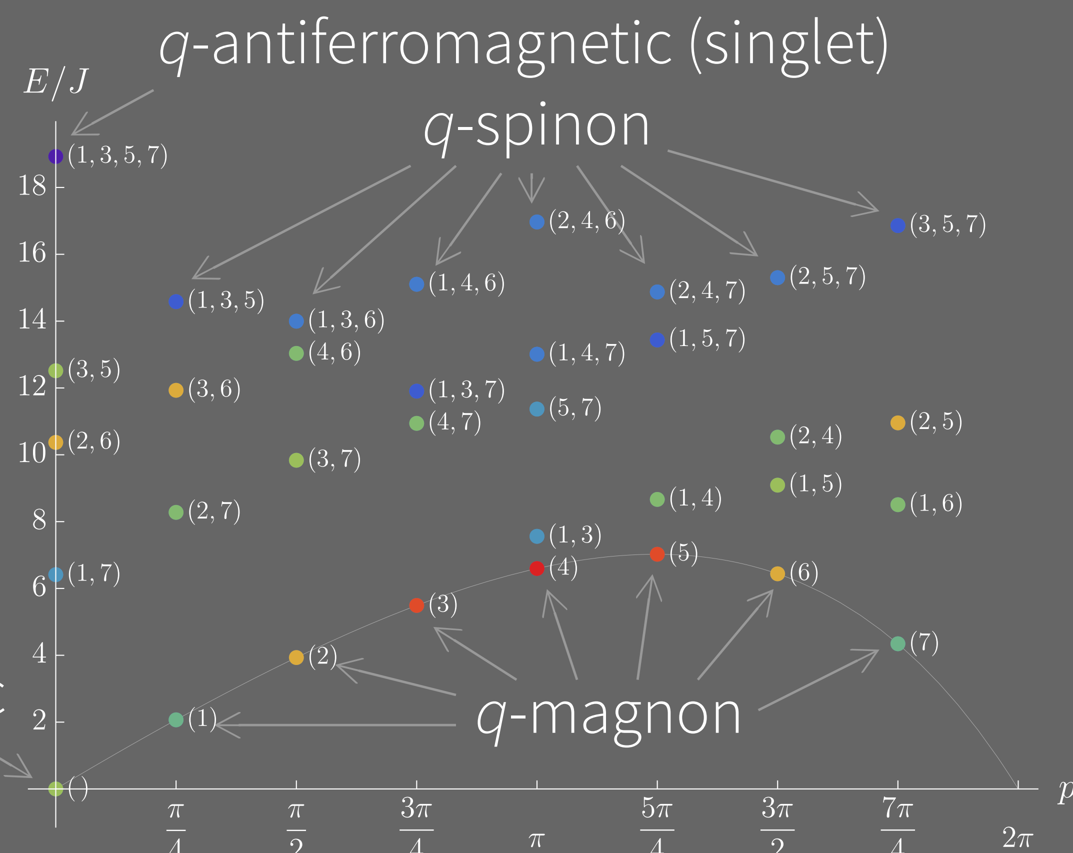


addition is allowed iff $m_{r+1} > m_r + 1$

$$p = \frac{2\pi}{L} \sum_r m_r \mod 2\pi$$

Exact spectrum at finite length
Interpretation of multiplets

$$E = \sum_r \varepsilon(m_r)$$



q-ferromagnetic

$(S_q^-)^M |\uparrow \cdots \uparrow\rangle$,

$0 \leq M \leq L$

addition is allowed iff

$m_{r+1} > m_r + 1$

cf 'generalized Pauli principle'

[Haldane '91]

$$p = \frac{2\pi}{L} \sum_r m_r \mod 2\pi$$

Finer structure

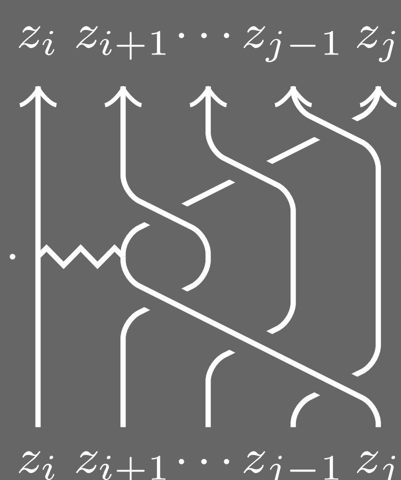
Relation with Temperley–Lieb algebra

[JL ’18]

- Since $[U_q(\mathfrak{sl}_2), H] = 0$ we have $H \in \mathrm{TL}_L(q+q^{-1})$
- In fact it is easily written via the generators:

$$z_j = \mathrm{e}^{2\pi \mathrm{i}\, j/L} \qquad q = \mathrm{e}^{\gamma}$$

$$H = -J \sum_{i<j}^{L} V(i-j)\, S_{[i,j]} \qquad V = \frac{z_i\, z_j}{(q\, z_i - q^{-1} z_j)(q\, z_j - q^{-1} z_i)}$$

$S_{[i,j]} = \cdots$  $\cdots$

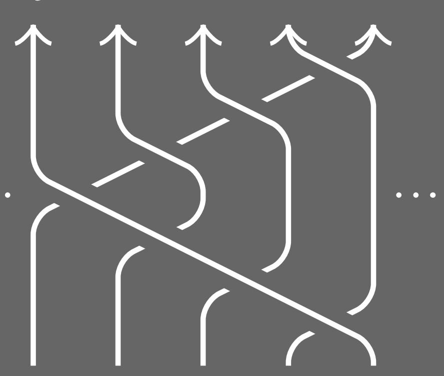
$$= \check{R}\Big(\frac{v}{u}\Big) = 1 - \frac{u-v}{q\,u - q^{-1} v}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

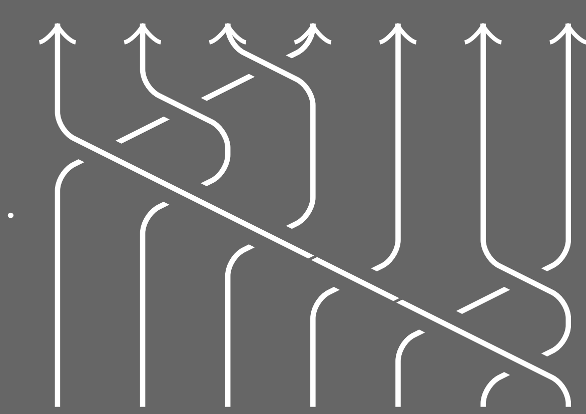
$$= (q - q^{-1})\, \check{R}'(1) = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Finer structure
Uglov's expression

$$\tilde{H} = -J \sum_{N=2}^{L} (-1)^N \sum_{i_1<\cdots<i_N} \tilde{V}(i_1,\cdots,i_N)\,(Y_{[i_1,\cdots,i_N]} - 1)$$

$$\tilde{V}(i_1,\cdots,i_N) = \frac{L\,[L]_q^{-1}}{(q-q^{-1})^2} \prod_{n=1}^{N} \left(\frac{(q-q^{-1})\,z_{i_{n+1}}}{z_{i_{n+1}} - z_{i_n}} \prod_{k=i_n+1}^{i_{n+1}-1} \frac{q^{-1}\,z_{i_n} - q\,z_k}{z_{i_n} - z_k} \right)$$

$N=2$: $Y_{[i,j]} = \cdots$  $\cdots$

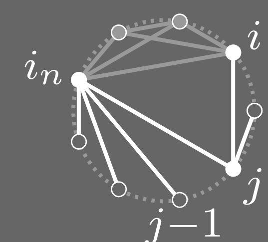
$N>2$: e.g. $Y_{[i_1,i_2,i_3,i_4]} = \cdots$  $\cdots$

Finer structure

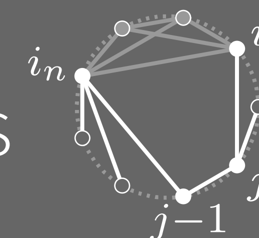
Sketch of proof of equality

[JL '18]

- Given $i = i_{\min}\,,\ j = i_{\max} > i+1$:



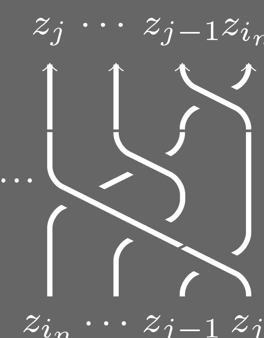
vs



$$h_{[i,j]} = \sum_{n=0}^{(j-1)-i} (-1)^n \sum_{i=i_0<\cdots<i_n<j-1} \Big[\tilde{V}(i_0,\cdots,i_n,j)\,(Y_{[i_0,\cdots,i_n]}\,Y_{[i_n,\,j]}-1) - \tilde{V}(i_0,\cdots,i_n,j-1,j)\,(Y_{[i_0,\cdots,i_n]}\,Y_{[i_n,\,j-1,\,j]}-1)\Big]$$

↑ # intermediate interacting spins (excluding $j-1$)

$Y_{[i_n,\,j]} = \cdots$



$\cdots = \check{R}_{j-1,\,j}\Big(\dfrac{z_{j-1}}{z_{i_n}}\Big) \times Y_{[i_n,\,j-1,\,j]}$

- Further ingredients:

$$\frac{q\,u-q^{-1}}{u-1}\,\big(\check{R}(u)-1\big) = (q-q^{-1})\,\check{R}'(1)$$

$$\sum_{N=2}^{L}(-1)^N \sum_{i_1<\cdots<i_N} \tilde{V}(i_1,\cdots,i_N) = \sum_{i<j}^{L} \frac{z_i\,z_j}{(q\,z_i-q^{-1}z_j)(z_j-z_i)}$$

Conclusion
Summary

- **Haldane–Shastry** model can be ***q*-deformed**
  - key properties: **infinitely many symmetries**, **additivity**
  - **pairwise form**, accounting for **multi-spin** interactions
- The **exact spectrum** at finite size is known
  - **dispersion relation** is ***q*-deformed**, so **level splitting**
  - **momentum**'s definition is ***q*-deformed**
  - **free quasiparticles** subject only to **statistics** (cf motifs)

Conclusion
Future directions

- More detail
  - highest-weight wave functions (~ Macdonald polynomials?)
  - precise (level $c = 0$) action of quantum-affine $\mathfrak{sl}_2$
  - (conformal?) field theory describing low energy
  - q root of unity ($q = \mathrm{e}^{\pi\mathrm{i}/2}, \mathrm{e}^{\pi\mathrm{i}/3}, \mathrm{e}^{\pi\mathrm{i}/4}, \dots$)
- Generalizations
  - higher rank: not just $\mathfrak{sl}_n$, but $\mathfrak{sl}_{n|m}$ (and other types)
  - q-deformed Inozemtsev spin chain
- …