Diagonalization and maximal tori

Let $n \geq_{\mathbb{Z}} 0$.

Definition 1. A compact Lie group G is a (Lie) n-torus iff $G \overset{\text{Lie group}}{\simeq} (\mathbb{R}/\mathbb{Z})^n$.

Remark 1.1. There is another often used definition of Lie torus equivalent to this one.

Let

$$\begin{aligned} T &:= \text{ group of diagonal matrices in } \mathrm{U}_n \\ &= \{\operatorname{diag}((e^{i\theta_l})_{l=1}^n) \mid (\theta_l)_{l=1}^n \in \mathbb{R}^n\}. \end{aligned}$$

If you know the closed-subgroup theorem you can easily show that T is a Lie group. Also T is a Lie torus (we skip checking the natural group isomorphism is a diffeomorphism).

Proposition 2. *T is a maximal abelian-subgroup of* U_n*.*

Proof. WLOG assume $n \geq 2$. Assume $\exists U \in \mathrm{C}_{\mathrm{U}_n}(T) \backslash T$. WLOG assume U has a non-0 entry in its strict upper triangle. Then we can write $U = \begin{pmatrix} *_m & V \\ * & * \end{pmatrix}$, where $V \neq 0$. We have $\begin{pmatrix} -I_m & \\ & I \end{pmatrix} \in T$. Then

$$\begin{aligned} \begin{pmatrix} -I_m & \\ & I \end{pmatrix} \begin{pmatrix} * & V \\ * & * \end{pmatrix} &= \begin{pmatrix} * & -V \\ * & * \end{pmatrix} \\ &\neq \begin{pmatrix} * & V \\ * & * \end{pmatrix} \\ &= \begin{pmatrix} * & V \\ * & * \end{pmatrix} \begin{pmatrix} -I_m & \\ & I \end{pmatrix}, \end{aligned}$$

a contradiction. $\square$

Let $\mathfrak{U} \subseteq \mathrm{M}_n\mathbb{C}$.

Definition 3. $\mathfrak{U}$ is reducible iff $\exists S \in \mathrm{GL}_n\mathbb{C}, \exists q \in \{r\}_{r=1}^{n-1}, \forall A \in \mathfrak{U}, S^{-1}AS = \begin{pmatrix} *_q & C \\ & * \end{pmatrix}$. $\mathfrak{U}$ is unitarily reducible iff S can be made $\in \mathrm{U}_n$. Reducible can be replaced with irreducible in the obvious way. $\mathfrak{U}$ is fully reducible iff C can be made 0. Unitarily fully reducible means what you think it means.

Lemma 4. *Let $\mathfrak{U}$ be reducible. Then $\mathfrak{U}$ is unitarily reducible.*

Proof. Let $S \in \mathrm{GL}_n\mathbb{C}$ be s.t. $\exists q \in \{r\}_{r=1}^{n-1}, \forall A \in \mathfrak{U}, S^{-1}AS = \begin{pmatrix} *_q & * \\ & * \end{pmatrix}$. Then by a theorem from previous talks $\exists U \in \mathrm{U}_n, \exists$upper triangular $R \in \mathrm{GL}_n\mathbb{C}, S = UR$. Then

$$\begin{aligned} U^{-1}AU &= RS^{-1}ASR^{-1} \\ &= \begin{pmatrix} *_q & * \\ & *_{n-q} \end{pmatrix} \begin{pmatrix} *_q & * \\ & *_{n-q} \end{pmatrix} \begin{pmatrix} *_q & * \\ & *_{n-q} \end{pmatrix} \\ &= \begin{pmatrix} *_q & * \\ & * \end{pmatrix}. \end{aligned}$$

□

A semigroup satisfies all groups axioms except for the inverse axiom. Centralizers can be defined for semigroups in the obvious way.

Lemma 5 (specialized Schur's lemma)**.** *Let $\mathfrak{U}$ be unitarily irreducible. Let $M \in \mathrm{C}_{\mathrm{M}_n\mathbb{C}}(\mathfrak{U})$. Then $\exists \lambda \in \mathbb{C}, M = \lambda I$.*

Proof. WLOG assume $n \geq 1$. Since M's characteristic polynomial has a root, we get M has an eigenvalue λ. Let $N := M - \lambda I$ & $r := \operatorname{rank} N < n$. Then $\exists P, Q \in \mathrm{GL}_n\mathbb{C}, N = P \begin{pmatrix} I_r & \\ & 0 \end{pmatrix} Q$. Write

$$\forall A \in \mathfrak{U}, P^{-1}AP = \begin{pmatrix} *_r & * \\ D & * \end{pmatrix}. \tag{5.1}$$

Then

$$AN = NA$$
$$AP \begin{pmatrix} I_r & \\ & 0 \end{pmatrix} Q = P \begin{pmatrix} I_r & \\ & 0 \end{pmatrix} QA$$

$$(P^{-1}AP)\begin{pmatrix} I_r & \\ & 0 \end{pmatrix} = \begin{pmatrix} I_r & \\ & 0 \end{pmatrix}(QAQ^{-1})$$
$$\begin{pmatrix} *_r & \\ D & 0 \end{pmatrix} = \begin{pmatrix} *_r & * \\ & 0 \end{pmatrix}$$
$$D = 0.$$

So $r = 0$ since otherwise $\mathfrak{U}$ would be reducible by (5.1) and so unitarily reducible by lemma 4. So $M - \lambda I = 0$. □

Lemma 6. *Let $\mathfrak{U}$ be unitarily reducible and only contain normal matrices. Then $\mathfrak{U}$ is unitarily fully reducible.*

Proof. We have $\exists U \in \mathrm{U}_n, \exists q \in \{r\}_{r=1}^{n-1}, \forall A \in \mathfrak{U}, \mathbb{C}, U^{-1}AU = \begin{pmatrix} B & C \\ & *_q \end{pmatrix}$. Then

$$\begin{aligned} U^{-1}AU(U^{-1}AU)^\dagger &= U^\dagger AUU^\dagger A^\dagger U \\ &= U^\dagger A^\dagger AU \\ &= U^\dagger A^\dagger UU^\dagger AU \\ &= (U^{-1}AU)^\dagger U^{-1}AU \\ \begin{pmatrix} B & C \\ & * \end{pmatrix}\begin{pmatrix} B^\dagger & \\ C^\dagger & * \end{pmatrix} &= \begin{pmatrix} B^\dagger & \\ * & * \end{pmatrix}\begin{pmatrix} B & * \\ & * \end{pmatrix} \\ \begin{pmatrix} BB^\dagger + CC^\dagger & * \\ * & * \end{pmatrix} &= \begin{pmatrix} B^\dagger B & * \\ * & * \end{pmatrix} \\ B^\dagger B - BB^\dagger &= CC^\dagger \\ 0 &= \mathrm{tr}(CC^\dagger) \\ &= \sum_{l=1}^{q}\sum_{m=1}^{n-q} |(C)_{l,m}|^2. \end{aligned}$$

So $C = 0$. □

Proposition 7. *Let $\mathfrak{U}$ be an abelian subgroup of U_n. Then $\exists S \in \mathrm{U}_n, S^{-1}\mathfrak{U}S \subseteq T$.*

Proof. If $\mathfrak{U}$ is unitarily reducible then by lemma 6 $\exists S_\emptyset \in \mathrm{U}_n, \exists q \in \{s\}_{s=1}^{n-1}$, $\forall A \in \mathfrak{U}, \exists M_{\emptyset,A} \in \mathrm{M}_q\mathbb{C}, S_\emptyset^{-1}AS_\emptyset = \begin{pmatrix} M_{\emptyset,A} & \\ & N_{\emptyset,A} \end{pmatrix}$. Let $\mathfrak{U}_0 := \{M_{\emptyset,A} \mid A \in \mathfrak{U}\}$.

We have $\forall A, B \in \mathfrak{U}$,

$$S_\emptyset^{-1} A S_\emptyset S_\emptyset^{-1} B S_\emptyset = S_\emptyset^{-1} B S_\emptyset S_\emptyset^{-1} A S_\emptyset$$
$$\begin{pmatrix} M_{\emptyset,A} M_{\emptyset,B} & \\ & N_{\emptyset,A} N_{\emptyset,B} \end{pmatrix} = \begin{pmatrix} M_{\emptyset,B} M_{\emptyset,A} & \\ & N_{\emptyset,B} N_{\emptyset,A} \end{pmatrix}.$$

So $\mathfrak{U}_0$ is commutative. Also

$$\begin{aligned} I &= S_\emptyset^\dagger A S_\emptyset S_\emptyset^\dagger A^\dagger S_\emptyset \\ &= S_\emptyset^{-1} A S_\emptyset (S_\emptyset^{-1} A S_\emptyset)^\dagger \\ &= \begin{pmatrix} M_{\emptyset,A} M_{\emptyset,A}^\dagger & \\ & N_{\emptyset,A} N_{\emptyset,A}^\dagger \end{pmatrix}, \end{aligned}$$

so $\mathfrak{U}_0 \subseteq \mathrm{U}_q$. If $\mathfrak{U}_0$ is unitarily reducible, then $\exists S_0 \in \mathrm{U}_q, \exists r \in \{s\}_{s=1}^{q-1}, \exists M_{0,A} \in \mathrm{M}_r\mathbb{C}$, $\begin{pmatrix} S_0^{-1} & \\ & I \end{pmatrix} S_\emptyset^{-1} A S_\emptyset \begin{pmatrix} S_0 & \\ & I \end{pmatrix} = \begin{pmatrix} M_{0,A} & & \\ & N_{0,A} & \\ & & N_{\emptyset,A} \end{pmatrix}$. Now repeat the whole process until we get an element S of U_n s.t. $S^{-1} A S = \mathrm{diag}((A_l)_{l=1}^t)$ and $\forall l \in \{s\}_{s=1}^t, \{A_l \mid A \in \mathfrak{U}\}$ is unitarily irreducible and commutative. Then by lemma 5 A_l is diagonal. □

Let X & Y be topological spaces.

Lemma 8. *Let X & Y both be connected. Then $X \times Y$ is connected.*

Proof. Let $F\colon X \times Y \to \{0, 1\}$ be continuous where $\{0, 1\}$ is discrete. Then $\forall x \in X$, $\begin{matrix} Y \to \{0,1\} \\ y \mapsto F(x,y) \end{matrix}$ is constant since Y is connected. Similarly, $\forall y \in Y$, $\begin{matrix} X \to \{0,1\} \\ x \mapsto F(x,y) \end{matrix}$ is constant. Then $\forall w \in X, \forall z \in Y$,

$$\begin{aligned} F(x, y) &= F(x, z) \\ &= F(w, z). \end{aligned}$$

□

Lemma 9. *Let S be an embedded C^∞ submanifold of U. Then* $\dim S = \dim U$ *iff $S \in \mathcal{T}_U$.*

Proof. See proposition 5.1 of [4, pp. 99] □

Proposition 10. *Let G be a compact Lie group. Then $\exists$maximal Lie torus $\subseteq G$.*

Proof. Start with $\{1_G\} \overset{\text{Lie group}}{\simeq} (\mathbb{R}/\mathbb{Z})^0$. Let $S \subsetneq U \subseteq G$ be s.t. S & U are Lie tori. S is compact and U is Hausdorff, so S is closed in U. Also U is connected by lemma 8 and $\mathbb{R}/\mathbb{Z}$ being connected. So $S \notin \mathcal{T}_U$ since otherwise $\{S, U \setminus S\}$ is an open partition of U. So by lemma 9 $\dim S < \dim U$. □

Definition 11. X is homotopy equivalent to Y iff $\exists f \in \mathrm{Cts}(X,Y), \exists g \in \mathrm{Cts}(Y,X)$, $f \circ g$ & $g \circ f$ are both homotopic to id. X is contractible iff it's homotopy equivalent to $\{*\}$.

Example 11.1. Let $\varphi\colon \begin{array}{c} D^n \to \{0\} \\ x \mapsto 0 \end{array}$ and $\psi\colon \begin{array}{c} \{0\} \to D^n \\ 0 \mapsto 0 \end{array}$. Then $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi\colon \begin{array}{c} D^n \to D^n \\ x \mapsto 0 \end{array}$. Let $H\colon \begin{array}{c} D^n \times [0,1] \to D^n \\ (x,\alpha) \mapsto \alpha x \end{array}$. So $\psi \circ \varphi$ is homotopic to id. So D^n is contractible.

Definition 12. A CW-subcomplex of a CW-complex W is a union of closed cells $\subseteq W$ that form a CW-complex structure.

Remark 12.1. Intersections & unions of CW-subcomplexes of W are CW-subcomplexes of W.

Let W be a CW-complex of type $(c_l)_{l=0}^m$. Informally the universe is the set of all objects one wishes to consider.

Proposition 13. *Let the universe be $\mathcal{U}$. Let χ be a topological invariant function from the class of finite CW-complexes in $\mathcal{U}$ to $\mathbb{Z}$ s.t. if X & Y are CW-subcomplexes of a finite CW-complex $\in \mathcal{U}$ then*

1. $\chi(\emptyset) = 0$,
2. $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$,
3. $\chi(X) = 1$ *if X is contractible.*

Then $\chi(W) = \sum_{l=0}^m (-1)^l c_l$.

Proof. $\chi(S^0) = 2$.

$$\begin{aligned}\chi(S^n) &= \chi(\{(x_l)_{l=0}^n \in S^n \mid x_n \geq 0\} \cup \{(x_l)_{l=0}^n \in S^n \mid x_n \leq 0\}) \\ &= \chi(D^n) + \chi(D^n) - \chi(S^{n-1})\end{aligned}$$

$$= 2 - \chi(S^{n-1}).$$

So by induction we get $\chi(S^n) = 1 + (-1)^n$. Let A be a closed m-cell $\subseteq W$. Then

$$\begin{aligned}\chi(W) &= \chi(A \cup (W \setminus \operatorname{int}_W A)) \\ &= \chi(D^m) + \chi(W \setminus \operatorname{int}_W A) - \chi(S^{m-1}) \\ &= (-1)^m + \chi(W \setminus \operatorname{int}_W A).\end{aligned}$$

Induction finishes the proof. □

Remark 13.1. Apparently χ exists.

Definition 13.2. $\chi(W)$ is the Euler number/Euler characteristic of W.

Theorem 14. *If $\chi(W) \neq 0$ then any $f : W \to W$ homotopic to* id *has a fixed point.*

Proof. See the Lefschetz fixed-point theorem. □

Lemma 15. *Let S be a subgroup of a topological group G. Then $\rho\colon G \to G/S$ is open.*

Proof. Let $U \in \mathcal{T}_G$. Then $\forall g \in G$,

$$\begin{aligned} & \rho(g) && \in \rho(U) \\ \iff \quad & \exists u \in U, gS && = uS \\ \iff \quad & g && \in US. \end{aligned}$$

So

$$\begin{aligned}\rho^{-1}(\rho(U)) &= US \\ &= \textstyle\bigcup_{s \in S} Us \\ &\in \mathcal{T}_G \qquad \text{(right multiplication by } s \text{ is a homeomorphism).}\end{aligned}$$ □

Lemma 16. *Let M be a connected topological n-manifold. Then M is path-connected.*

Proof. Let $x, z \in M$ & $U := \{y \in M \mid \exists \text{path from } x \text{ to } y\}$. Then $(\exists V \in \mathcal{T}_M)(z \in V \wedge V \overset{\text{topology}}{\simeq} \mathbb{R}^n)$. If $z \in U$ then $V \subseteq U$ so $U \in \mathcal{T}_M$. If $z \in M \setminus U$ then $V \subseteq M \setminus U$ since otherwise $z \in U$. Hence $M \setminus U \in \mathcal{T}_M$. So $U = M$ since $x \in U$. □

Let B be the group of all upper triangular matrices $\in \mathrm{GL}_n\mathbb{C}$.

Theorem 17. $\mathrm{U}_n/T \overset{\text{topology}}{\simeq} (\mathrm{GL}_n\mathbb{C})/B$.

Proof. See a previous talk. □

Theorem 18. *$(\mathrm{GL}_n\mathbb{C})/B$ can be decomposed into cells $\overset{\text{topology}}{\simeq}$ $\mathbb{Z}_{\geq 0}$-powers of $\mathbb{C}$.*

Proof. See a previous talk. □

Corollary 19. *U_n/T can be decomposed into even-dimensional cells.*

Proof. Immediate from theorem 17 & theorem 18. □

Theorem 20. *Let S be a maximal Lie torus in a compact connected Lie group G, $g \in G$, & A be a connected abelian subgroup of G. Then*

1. $\exists y \in G, y^{-1}gy \in S$, &
2. $\exists y \in G, y^{-1}Ay \subseteq S$.

Proof idea. Since $\rho\colon G \to G/S$ & $\varphi\colon \begin{matrix} G \times G \to G \\ (x,y) \mapsto xy \end{matrix}$ are continuous, we get $\rho \circ \varphi$ is continuous. By lemma 15 $\mathrm{id} \times \rho$ is open. Let $\psi\colon \begin{matrix} G \times (G/S) \to G/S \\ (x,yS) \mapsto xyS \end{matrix}$ & $V \in \mathcal{T}_{G/S}$. Then

$$\begin{aligned}\psi^{-1}(V) &= \{(x,yS) \in G \times (G/S) \mid xyS \in V\} \\ &= (\mathrm{id} \times \rho)(\{(x,y) \in G \times G \mid xyS \in V\}) \\ &= (\mathrm{id} \times \rho)(\rho \circ \varphi)^{-1}(V) \\ &\in \mathcal{T}_{G \times (G/S)}.\end{aligned}$$

So ψ is continuous. Let $\Psi\colon \begin{matrix} \mathrm{Cts}(G \times (G/S), G/S) \to \mathrm{Cts}(G, \mathrm{Cts}(G/S, G/S)) \\ \varphi \mapsto (x \mapsto (Y \mapsto \varphi(x,Y))) \end{matrix}$ & $f_g := \Psi(\psi)(g)$. Since G is connected, we get G is path-connected by lemma 16. So f_g is homotopic to id. In corollary 19 apparently U_n/T can be replaced with G/S. So by theorem 14 $\exists y \in G$,

$$yS = f_g(yS)$$

$$
\begin{aligned}
&= gyS \\
S &\ni y^{-1}gy.
\end{aligned}
$$

Also $\exists a \in A, \mathrm{cl}_G(\{a^m \mid m \geq_{\mathbb{Z}} 0\}) = A$ (see proposition 4.4 of [2, pp. 80]) so $y^{-1}ay \in S \implies y^{-1}Ay \subseteq S$ since S is closed in G. $\square$

Remark 20.1. Apparently from G/S's Bruhat decomposition we can deduce $\chi(G/S) = \mathrm{ord}(\text{the Weyl group of } G \text{ wrt } S)$. This equality is also proven in [2, pp. 90–92].

Example 20.2. Let $S := \left\{ \begin{pmatrix} 1 & \\ & A \end{pmatrix} \middle| A \in \mathrm{SO}_2\mathbb{R} \right\}$. Let $\begin{pmatrix} *_1 & B \\ C & * \end{pmatrix} \in \mathrm{C}_{\mathrm{SO}_3\mathbb{R}} S$. Then

$$
\begin{pmatrix} * & B \\ C & * \end{pmatrix} \begin{pmatrix} 1 & \\ & -I \end{pmatrix} = \begin{pmatrix} 1 & \\ & -I \end{pmatrix} \begin{pmatrix} * & B \\ C & * \end{pmatrix}
$$
$$
\begin{pmatrix} * & -B \\ C & * \end{pmatrix} = \begin{pmatrix} * & B \\ -C & * \end{pmatrix}
$$

so $B = C = 0$. So $\mathrm{C}_{\mathrm{SO}_3\mathbb{R}} S \subseteq 2$ copies of $\mathbb{R}/\mathbb{Z}$ and so S is a maximal Lie torus in $\mathrm{SO}_3\mathbb{R}$. Let

$$
\begin{aligned}
A :=& \text{ group of diagonal matrices in } \mathrm{SO}_3\mathbb{R} \\
=& \left\{ I, \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \right\}.
\end{aligned}
$$

Then A has 3 elements of order 2 so A is not conjugate to a subgroup of S.

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