Last time

Gelfand Tsetlin algebra.

$$\{1\} = G(0) \subsetneq G(1) \cdots \subsetneq G(n).$$

$$\mathbb{C}[G(0)] \subseteq \mathbb{C}[G(1)] \subsetneq \cdots \subsetneq \mathbb{C}[G(n)]$$
$$\begin{array}{ccc} \cup\!\shortparallel & & \cup\!\shortparallel \\ Z(1) & \cdots & Z(n) \end{array}$$

Define

$$GZ(n) = \langle Z(1), Z(2), \ldots, Z(n) \rangle$$

Example

$$\mathbb{C}[S_n].$$

Gelfand Tsetlin basis.

Let simple branching $\leftarrow \dim(\mathrm{Hom}_{G(m)}(V^\mu, V^\lambda)) \in \{0,1\}$.

$$\rho : \mathbb{C}[G(n)] \longrightarrow \mathrm{End}(V^{\lambda^{(n)}}).$$

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Now

$$\mathrm{Res}^{G(n)}_{G(n-1)} V^{\lambda(n)} = \bigoplus_{\mu \nearrow \lambda} V^\mu.$$

and

$$\mathrm{Res}^{G(n)}_{\substack{G(0) \\ \shortparallel \\ \{1\}}} V^{\lambda(n)} = \bigoplus_T V_T.$$

GZ basis is

$\mathbb{C}$-span $\{v_T\}_T$ is a path in the branching graph.

Now

② $\Rightarrow$ ③.

Since
$$GZ(n) = \langle Z(1), Z(2), \ldots, Z(n) \rangle$$
$$= \langle c^{\ydiagram{2}}, c^{\ydiagram{2,1}}, c^{\ydiagram{2,1,1}}, \ldots, c^{\ydiagram{2,1,1,1}} \rangle.$$

and
$$c^{\ydiagram{2}} = X_2 = (1,2)$$
$$c^{\ydiagram{2,1}} = X_2 + X_3 = (1,2) + (1,3) + (2,3).$$
$$c^{\ydiagram{2,1,1}} = X_2 + X_3 + X_4 = (12) + (13) + (23) + (1,4) + (2,4) + (3,4),$$

so
$$GZ(n) = \langle X_1, X_2, \ldots, X_n \rangle. \quad \text{Note } X_2^2 = Id, \; X_1 = 0.$$

$\leftarrow$ commutes with each other.

Define
$$X_i V_T = c(T(i)) V_T.$$

$\uparrow$ Gelfand basis.

Let $V_{\ytableaushort{123,4}}$. Then

$$X_1 V_{\ytableaushort{123,4}} = c(T(1)) V_T = 0.$$

Recall
~~Let~~ $V = \{ V_{\ytableaushort{1234}}, V_{\ytableaushort{123,4}}, V_{\ytableaushort{124,3}}, V_{\ytableaushort{134,2}}, V_{\ytableaushort{12,34}}, V_{\ytableaushort{13,24}},$
$V_{\ytableaushort{12,3,4}}, V_{\ytableaushort{13,2,4}}, V_{\ytableaushort{14,2,3}}, V_{\ytableaushort{1,2,3,4}} \}.$

Now $\varphi: \mathbb{C}[S_4] \longrightarrow \mathrm{End}(V) = V^{(4)} \oplus V^{(3,1)} \oplus V^{(2,2)} \oplus V^{(2,1,1)} \oplus V^{(1,1,1,1)}$

$$\pi(X_1) = \mathrm{diag}(0,0,\dots,0).$$

$$\pi(X_2) = \mathrm{diag}(1,1,1,-1,1,-1,1,-1,-1,-1)$$

(Arun Ram's Notes)

$$\pi(X_3) = \mathrm{diag}(2,2,-1,1,-1,1,-1,1,-2,-2)$$

$$\pi(X_4) = \mathrm{diag}(3,-1,2,2,0,0,-2,-2,1,-3).$$

$$\pi(C^{(2,2)}) = \mathrm{diag}(3,3,0,0,0,0,0,0,-3,-3)$$

$$\pi(C^{(2,1,1)}) = \mathrm{diag}(6,2,2,2,0,0,-2,-2,-2,-6)$$

$$\pi(Z_1) = \mathrm{diag}(1,0,0,\dots,0)$$

$$\pi(Z_2) = \mathrm{diag}(0,1,1,1,0,0,\dots,0)$$

$$\pi(Z_3) = \mathrm{diag}(0,0,0,0,1,1,0,\dots,0)$$

$$\pi(Z_4) = \mathrm{diag}(0,0,0,0,\dots,1,1,1,0).$$

$$\pi(Z_5) = \mathrm{diag}(0,0,0,\dots,0,1)$$

Note $V = V^{(4)} \oplus V^{(3,1)} \oplus V^{(2,2)} \oplus V^{(2,1,1)} \oplus V^{(1,1,1,1)}$

Hence, $\mathbb{C}[S_4] \cong \mathrm{End}(V^{(4)}) \oplus \mathrm{End}(V^{(3,1)}) \oplus \cdots$

$$\mathbb{C}[S_4] = \begin{pmatrix} a & & & & & & & & & \\ & b_{11} & b_{12} & b_{13} & & & & & & \\ & b_{21} & b_{22} & b_{23} & & & & & & \\ & b_{31} & b_{32} & b_{33} & & & & & & \\ & & & & c_{11} & c_{12} & & & & \\ & & & & c_{21} & c_{22} & & & & \\ & & & & & & d_{11} & d_{12} & d_{13} & \\ & & & & & & d_{21} & d_{22} & d_{23} & \\ & & & & & & d_{31} & d_{32} & d_{33} & \\ & & & & & & & & & e \end{pmatrix}$$

So

$$Z(\mathbb{C}[S_4], \mathbb{C}[S_3]) = \{ \operatorname{diag}(a_1, a_2, b_1, b_1, b_2, b_2, b_3, b_3, c_1, c_2) \}.$$ ④

If $m \in Z(\mathbb{C}[S_4], \mathbb{C}[S_3])$, then
$n \in \mathbb{C}[S_3]$

$$\rho(m)\big|_{V^{(3,1)}} \rho(n)\big|_{V^{(3,1)}} = \rho(n)\big|_{V^{(3,1)}} \rho(m)\big|_{V^{(3,1)}}$$

Schur's lemma $\Longrightarrow$

$$\rho(m)\big|_{V^{(3,1)}} = \left(\begin{array}{c|cc} a & & \\ \hline 0 & b & 0 \\ & 0 & b \end{array}\right)$$

$$Z(\mathbb{C}[S_4]) = \{ \operatorname{diag}(a, b, b, b, c, c, d, d, d, e) \}.$$

$$Z(\mathbb{C}[S_3]) = \{ \operatorname{diag}(a, a, b, b, b, b, b, b, c, c) \}.$$

Branching graph $\Leftrightarrow$ Young's lattice.

Recall Pieri's formula

$$s_\lambda h_r = \sum_{\mu} s_\mu \qquad h_r = \sum_{|\lambda| = r} m_\lambda.$$

where $\mu - \lambda$ is a horizontal r-strips.

Define

$$R = \bigoplus_{n \geq 0} R^n \longleftarrow \mathbb{Z}\text{-module } \{ \chi^\lambda \mid |\lambda| = n \}.$$

↑ irreducible character.

Multiplication if $f \in R^m$ and $g \in R^n$. Then

$$f \cdot g = \operatorname{Ind}_{S_m \times S_n}^{S_{m+n}} f \times g.$$

Define $f, g \in R^n$

$$\langle f, g \rangle_{S_n} = \frac{1}{|S_n|} \sum_{g' \in G} f(g') g(g') = \frac{1}{n!} \sum_{g' \in G} f(g') g(g').$$

and

$$\begin{aligned} \psi : S_n &\longrightarrow \Lambda_n \\ \omega &\longmapsto p_{\rho(\omega)} \end{aligned}$$

where $\rho(\omega)$ is the cycle type of ω.

Then

$$\begin{aligned} \mathrm{ch} : R &\longrightarrow \Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}. \\ f &\longmapsto \langle f, \psi \rangle_{S_n}. \end{aligned}$$

$\longleftarrow$ isomorphism.

$$\mathrm{ch}(f) = \langle f, \psi \rangle_{S_n} = \frac{1}{n!} \sum_{g \in S_n} f(g) \psi(g^{-1}) = \frac{1}{n!} \sum_{g \in S_n} f(g) \psi(g).$$

$$= \sum_{|\rho| = n} z_\rho^{-1} f_\rho p_\rho.$$

Recall $\mathrm{ch}(\chi^\lambda) = s_\lambda$.

Now $s_\lambda s_1 = \sum_\mu s_\mu$ $\quad \mu - \lambda$ is a box.

$$\mathrm{ch}(\chi^\lambda)\,\mathrm{ch}(\chi^\square) = \sum_\mu \mathrm{ch}(\chi^\mu) = \mathrm{ch}\Big(\sum_\mu \chi^\mu\Big)$$

$$\mathrm{ch}(\chi^\lambda \cdot \chi^\square) = \mathrm{ch}\Big(\sum_\mu \chi^\mu\Big)$$

$$\Rightarrow \quad \chi^\lambda \cdot \chi^\square = \sum_\mu \chi^\mu.$$

$$\Rightarrow \operatorname{Ind}_{S_{n-1}\times S_1}^{S_n} V^\lambda \otimes V^{\square} = \bigoplus_\mu V^\mu .$$

$$\Rightarrow \operatorname{Ind}_{S_{n-1}}^{S_n} V^\lambda = \bigoplus_\mu V^\mu \qquad \mu-\lambda \text{ is a box.}$$

Now restriction rule and induction rule ~~differ~~ are equivalent by Frobenius reciprocity

$$\langle \operatorname{Ind}_{S_{n-1}}^{S_n} V^\lambda, V^\mu \rangle = \langle V^\lambda, \operatorname{Res}_{S_{n-1}}^{S_n} V^\mu \rangle .$$