Mixed geometry and Kazhdan-Lusztig

$$\begin{tikzcd}
\mathrm{Mod}(U(\mathfrak{g})) \supset \mathrm{Mod}(U(\mathfrak{g}),N)^{\mathrm{fin}}_{\tilde{0}} \arrow[d, "\wr"] & \supset \mathrm{Mod}(U(\mathfrak{g}),B)^{\mathrm{fin}}_{0} \arrow[d, "\wr"] \\
\mathcal{D}\text{-mod}(G/N, N)^{\mathrm{fin}}_{\tilde{0}} & \supset \mathcal{D}\text{-mod}(G/B, B)^{\mathrm{fin}} \\
\mathrm{MHM}(G/N, N)^{\mathrm{fin}}_{\tilde{0}} \arrow[u] \arrow[dr, "K_0"'] & \supset \mathrm{MHM}(G/B, B)^{\mathrm{fin}} \arrow[u] \arrow[d, "K_0"] \\
& \mathcal{H} \otimes_{\mathbb{Z}[q,q^{-1}]} \overbrace{K_0(\mathrm{MHM}(pt))}^{\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]}
\end{tikzcd}$$

– finite length

$\supset \mathcal{O}_0 \supset$

Beilinson-Bernstein

$$q \longleftrightarrow t_1 t_2 = (-1)$$

Quotient: set $t_1 = t_2 = q^{1/2}$

Pf of KL: chase Verma modules and irreducibles through this diagram.

$$M_w = M(w\cdot(-2\rho)). \quad L_w = L(w\cdot(-2\rho))$$

$$-2\rho = w_0 \cdot 0$$

$j_w : X_w = BwB/B \hookrightarrow G/B$

$$\begin{tikzcd}
M_w \arrow[r] \arrow[d, mapsto] & L_w \arrow[r, hook] \arrow[d, mapsto] & M_w^\vee \arrow[d, mapsto] \\
j_{w!}\mathcal{O}_{X_w} \arrow[r] & j_{w!*}\mathcal{O}_{X_w} \arrow[r, hook] & j_{w*}\mathcal{O}_{X_w} \\
{} \arrow[u, mapsto] \arrow[d, mapsto] & \text{''} \arrow[u, mapsto] \arrow[d, mapsto] & {} \arrow[u, mapsto] \arrow[d, mapsto] \\
(-1)^{\ell(w)} T_w & (-1)^{\ell(w)} q^{\ell(w)/2} C_w & (-1)^{\ell(w)} q^{\ell(w)} \overline{T}_w
\end{tikzcd}$$

$IC(\overline{X}_w) = j_{w!*}\mathcal{O}_{X_w}$

Brylinski-Kashiwara

Hecke relation: $(T_s+1)(T_s-q) = 0$

Kazhdan-Lusztig polys

Bar involution $\beta(T_s) = T_s^{-1}$

$\beta(q^{1/2}) = q^{-1/2}$

Upgrade: $\beta(t_1) = t_2^{-1}$, $\beta(t_2) = t_1^{-1}$

C_w unique basis satisfying

$\beta(C_w) = C_w$ and

$$C_w = q^{-\ell(w)/2} \sum_{y \leq w} P_{y,w}(q) T_y$$

$P_{w,w} = 1$,
$P_{y,w}(q) \in \mathbb{Z}[q]$,
$\deg \leq \ell(w) - \ell(y) - 1$
if $y \neq w$

This gives:

$$\underbrace{[j_{w!*} \mathcal{O}_{X_w}]}_{[L_w]} = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} P_{y,w}(q) \underbrace{[j_{w!} \mathcal{O}_{X_w}]}_{[M_w]}$$

in $K_0(MHM(G/B, B)^{An})$

$q=1 \Rightarrow$ KL Conjecture

Details:

* What are all the categories?
* How do you get the iso with Hecke algebra + images of objects?

Notation:

$(\ )^{fn}$ = finite length objects

$\mathrm{Mod}(U(\mathfrak{g}), N)$ = $U(\mathfrak{g})$-modules with compatible N-action

$\mathrm{Mod}(U(\mathfrak{g}), N)_0$ = — " — with central char 0

$\mathrm{Mod}(U(\mathfrak{g}), N)_{\tilde{0}}$ = — " — generalized — " — N-equivariant

$D\text{-mod}(G/N, N)_{\tilde{0}}$ = D-modules on G/N, local systems on fibres of $G/N \to G/B$ st. monodromy is unipotent.

$\Leftrightarrow$ Monodromic D-modules

Pull back

$D\text{-mod}(G/B, B)$ = B-equivariant D-modules on G/B

MHM = same for mixed Hodge modules (purely complex version)

* Need big cat for Koszul duality + Jantzen?
* Small category is easier + captures Hecke product

How to get comparison w/ Hecke algebra?

$MHM(G/B, B)$ has convolution product:

$$\begin{tikzcd} G/B \times G/B & G \times G/B \arrow[l, "p"'] \arrow[r, "q"] & G \times^B G/B \arrow[r, "m"] & G/B \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F} \boxtimes \mathcal{G} \arrow[r, mapsto] & p^*\mathcal{F} \boxtimes \mathcal{G} \arrow[r, mapsto, "(q^*)^{-1}"] & \arrow[r, mapsto, "m_*"] & \mathcal{F} * \mathcal{G} \end{tikzcd}$$

$\circlearrowleft$ $B \times B$ $\qquad$ $\circlearrowleft$ $B \times B$ $\rightsquigarrow$ $\circlearrowleft$ B $\qquad$ $\circlearrowleft$ B

act diagonally

$\therefore K_0(MHM(G/B, B))$ is a $K_0(MHM(pt))$-algebra.

<u>Note:</u> $K_0(MHM(pt)) = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$

$$H = \bigoplus_{p,q} H^{p,q} \longmapsto \sum_{p,q} \dim H^{p,q}\, t_1^p t_2^q$$

$\{[j_{w!}\mathcal{O}_{X_w}]\}$ basis for $K_0(MHM(G/B, B))$

check (straightforward computation):

$$[j_{w!}\mathcal{O}_{X_w}] \longmapsto (-1)^{\ell(w)} T_w \quad \text{alg hom}$$

$$K_0(MHM(pt)) \xrightarrow{\sim} \mathcal{H} \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$$

Also have "Verdier" duality

$\mathbb{D} : MHM(G/B, B)^{op} \xrightarrow{\sim} MHM(G/B, B)$

induces β on K_0 (another computation)

$\mathbb{D}(M) = R\underline{Hom}(M, D_X)$

Have:

- $\mathbb{D}(j_{w!*}\mathcal{O}_{X_w}) = j_{w!*}\mathcal{O}_{X_w}(\ell(w))$

$\therefore \beta[j_{w!*}\mathcal{O}_{X_w}] = q^{-\ell(w)}[j_{w!}\mathcal{O}_{X_w}]$

Also

$$[j_{w!*}\mathcal{O}_{X_w}] = \sum_{y \leq w} Q_{w,y}(t_1,t_2)[j_{y!}\mathcal{O}_{X_y}]$$

for some $Q_{w,y}(t_1,t_2) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$

with $Q_{w,w} = 1$.

<u>In fact:</u> $Q_{w,y}(t_1,t_2) = [j_y^* \underbrace{j_{w!*}\mathcal{O}_{X_w}}_{\text{pure of wt } \ell(w)}]$

<u>Recursivity:</u> if $y \neq w$:

Cohomology in in degs $[-\ell(w), -\ell(y)-1]$

Joint degree in $\mathbb{Q}_{w,y}$ is $[0, \ell(w)-\ell(y)-1]$

In fact,

Bott-Samelson resolution

$$j_{w!*}\mathcal{O}_{X_w} \overset{\oplus}{\hookrightarrow} \pi_{\underline{w}*}\mathcal{O}_{\tilde{X}_{\underline{w}}}$$

So

$$j_y^* j_{w!*}\mathcal{O}_{X_w} \overset{\oplus}{\hookrightarrow} \left(\begin{array}{c}\text{Cohomology of} \\ \text{Bott-Samelson} \\ \text{fibre}\end{array}\right)$$

This has a cell decomposition by affine spaces

$\Rightarrow$ Cohomology is in degrees (p,p), $p \in \mathbb{Z}$.

So in fact $\mathbb{Q}_{w,y}$ poly in q.

$\Rightarrow$ Must be KL polynomials! (up to sign)