

26 Nov 2020

kari

①

Preliminary constructions.

$$\begin{array}{ccc}
 F & & 0 \\
 \downarrow j & & \downarrow \\
 X & \xrightarrow{f} & \mathbb{C} \\
 \uparrow j & & \uparrow \\
 U & \longrightarrow & \mathbb{C}^*
 \end{array}$$

We fix M hol on U .

$$M_0 \subseteq j_* M, \text{ } \mathcal{O}_X\text{-coherent s.t.}$$

$$D_X M_0 = j_* M$$

Using \mathcal{G} -filtration lemma

$\mathcal{D}f^s$ a \mathcal{D} -module on \mathbb{C} or \mathbb{C}^*

f^s pullback.

$$j_! f^s M(\{s\}) \xrightarrow{\cong} j_* f^s M(\{s\}) / \mathbb{C}(\{s\})$$

$$j_! f^s M(\{s\}) \hookrightarrow j_* f^s M(\{s\}) / \mathbb{C}(\{s\})$$

$$j_! * = \text{Im}(j_! \rightarrow j_*).$$

$$\text{General, } j_! * M = D_X(f^k M_0) \text{ for } k \gg 0.$$

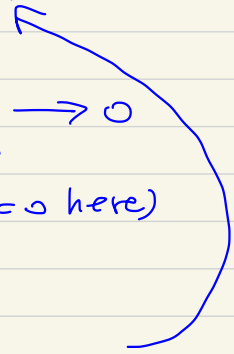
$$j_* M = D_X(f^{-k} M_0) \text{ for } k \gg 0.$$

$$\Rightarrow j_! f^S M(\mathbb{C}[s]) = D_x(\mathbb{C}[s]) (f^{st+k} M_0), \text{ for } k \gg 0.$$

$$j_! M = D_x(\mathbb{C}[s]) (f^{st+k} M_0) / s D_x(\mathbb{C}[s]) (f^{st+k} M_0) \quad \text{for } k \gg 0$$

$$0 \rightarrow f^S M(\mathbb{C}[s]) \xrightarrow{s} f^S M(\mathbb{C}[s]) \rightarrow M \rightarrow 0$$

↑
(s=0 here)



Apply $j_!$ to this exact sequence, obtain.
(exact)

On the disk, we can think of the module

$$\mathcal{J}^{(n)} = \mathcal{O}_{\mathbb{C}^*} \otimes \mathbb{C}[s]/s^n \quad \text{generated by } t^s \quad (t \text{ coordinate on } \mathbb{C}).$$

$$\mathcal{J}^\infty = \mathcal{O}_{\mathbb{C}^*}[[s]]. \quad \text{let } M \otimes f^* \mathcal{J}^{(n)} =: M(n).$$

$$\mathcal{J}^{(n)} : j_! f^S M(n) \xrightarrow{s^a} j_* f^S M(n) \quad a \geq 0.$$

Claim: $\text{Coker}(S^{\infty} \text{cn})$ stabilizes by the \mathfrak{b} -function lemma.

(3)

Def: $\text{Cok}(S^{\infty} \text{cn}) = \underbrace{\Pi f^{\infty} M}_{\text{Beilinson functor}}, n \gg 0.$

"pf": Have to show that $\text{Cok } S^{\infty}(\infty)$ is S -torsion.

$$j_* f^S M[[S]] = D_x[[S]] (f^{S+k} M_0) \quad k \gg 0$$

$$j_! f^S M[[S]] = D_x[[S]] (f^{S+k} M_S) \quad k \gg 0.$$

Take \mathfrak{b} -functions for generators of M_0 . Can assume M_0 is generated by one element u .

$$P(f^{S+1} u) = \mathfrak{b}(S) f^S u$$

$$P^{2k}(f^{S+k} u) = \underbrace{\mathfrak{b}(S+k-1) \cdots \mathfrak{b}(S+k)}_{\tilde{\mathfrak{b}}(S)} f^{S+k} u$$

$f^{S+k} u$ generates $j_* f^S M[[S]] / j_! f^S M[[S]]$.

Now $\tilde{\mathfrak{b}}(S) f^{S-k} u = P^{2k} f^{S+k} u \in j_! f^S M[[S]]$.

Thus, $\tilde{\mathfrak{b}}(S)$ annihilates $j_* f^S M[[S]] / j_! f^S M[[S]]$. \square

Fact: Π_f^a is an exact functor: $\text{Hol}(U) \rightarrow \text{Hol}(X)$.

$\Pi_f^o = \Psi_f^{un}$: Unipotent nearby cycles $\text{Hol}(U) \rightarrow \text{Hol}(F)$.

$\Pi_f^1 = \Sigma_f$ "maximal extension".

Rmk: To keep track of Hodge filtrations one should define nearby cycle functor via V -filtration.

