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Preliminary constructions.

$$\begin{array}{ccc} F & & 0 \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xrightarrow{f} & \mathbb{C} \\ {\scriptstyle j}\uparrow & & \uparrow \\ U & \longrightarrow & \mathbb{C}^* \end{array}$$

We fix M hol on U.

$M_0 \subseteq j_* M$, $\mathcal{O}_X$-coherent s.t.

$D_X M_0 = j_* M$

Using G-filtration lemma

$$j_! f^s M((s)) \xrightarrow{\simeq} j_* f^s M((s)) / \mathbb{C}((s))$$

$$\uparrow$$

$$j_! f^s M[[s]] \hookrightarrow j_* f^s M[[s]] / \mathbb{C}[[s]]$$

Df^s a D-module on $\mathbb{C}$ or $\mathbb{C}^*$

f^s pullback.

$j_{!*} = \mathrm{Im}(j_! \to j_*)$.

General, $j_{!*} M = D_X(f^k M_0)$ for $k \gg 0$.

$j_* M = D_X(f^{-k} M_0)$ for $k \gg 0$.

$\Rightarrow \quad j_! f^s \mathcal{M}[[s]] = D_X[[s]]\,(f^{s+k}\mathcal{M}_0)$ for $k >> 0$.

$j_! \mathcal{M} = D_X[s]\,(f^{s+k}\mathcal{M}_0) \,/\, s\, D_X[s]\,(f^{s+k}\mathcal{M}_0)$ for $k >> 0$

$$0 \to f^s\mathcal{M}[[s]] \xrightarrow{s} f^s\mathcal{M}[[s]] \to \mathcal{M} \to 0$$

($s = 0$ here)

Apply $j_!$ (exact) to this exact sequence, obtain.

On the disk, we can think of the module

$J^{(n)} = \mathcal{O}_{\mathbb{C}^*} \otimes \mathbb{C}[s]/s^n$ generated by t^s (t coordinate on $\mathbb{C}$).

$J^{\infty} = \mathcal{O}_{\mathbb{C}^*}[[s]]$. Let $\mathcal{M} \otimes f^* J^{(n)} =: \mathcal{M}(n)$.

$$\beta^a(n) : j_! f^s\mathcal{M}(n) \xrightarrow{s^a} j_* f^s\mathcal{M}(n) \qquad a \geq 0.$$

Claim: $\mathrm{Coker}(s^a(n))$ stabilizes by the b-function lemma.

Def: $\mathrm{Coker}(s^a(n)) = \Pi_f^a \mathcal{M}$, $n \gg 0$.

Π_f^a: Beilinson functor

"pf": Have to show that $\mathrm{Cok}\, s^a(\infty)$ is s-torsion.

$$j_* f^s \mathcal{M}[[s]] = \mathcal{D}_X[[s]](f^{s-k}\mathcal{M}_0) \qquad k \gg 0$$

$$j_! f^s \mathcal{M}[[s]] = \mathcal{D}_X[[s]](f^{s+k}\mathcal{M}_0) \qquad k \gg 0.$$

Take b-functions for generators of $\mathcal{M}_0$. Can assume $\mathcal{M}_0$ is generated by one element u.

$$P(f^{s+1}u) = b(s) f^s u$$

$$P^{2k}(f^{s+k}u) = \underbrace{b(s+k-1)\cdots b(s-k)}_{\tilde{b}(s)} f^{s-k}u$$

$f^{s-k}u$ generates $j_* f^s \mathcal{M}[[s]] / j_! f^s \mathcal{M}[[s]]$.

Now $\tilde{b}(s) f^{s-k}u = P^{2k} f^{s+k}u \in j_! f^s \mathcal{M}[[s]]$. Thus, $\tilde{b}(s)$ annihilates $j_* f^s \mathcal{M}[[s]] / j_! f^s \mathcal{M}[[s]]$. ☐

Fact: Π_f^a is an exact functor : $\mathrm{Hol}(U) \to \mathrm{Hol}(X)$.

$\Pi_f^0 = \psi_f^{un}$: unipotent nearby cycles $\mathrm{Hol}(U) \to \mathrm{Hol}(F)$.

$\Pi_f^! = \Xi_f$ "maximal extension."

Rmk: To keep track of Hodge filtrations one should define nearby cycle functor via V-filtration.