INTERLUDE

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1. Twisted sheaves

Let G be a (connected) reductive group over $\mathbb{C}$. Let $N \subset B \subset G$ a Borel and its unipotent radical. Let us consider the base affine space $\tilde{X} = G/N$ and the flag manifold $X = G/B$ and let us write $H = B/N$ for the universal Cartan. The map $\pi : \tilde{X} \to X$ is an H-fibration with H acting on $\tilde{X}$ via $h \cdot \tilde{x} = \tilde{x}h^{-1}$.

Let $T \subset G$ a maximal torus. Choosing a fixed point $x \in X^T$, and writing B_x for the Borel corresponding to x we get a map

$$\tau_x : T \to B_x \to H$$

which lets us identify T with H. We will use convention that the roots in B_x are negative and in this way H is equipped with the system of positive roots. It is not difficult to see that the system of positive roots on H in independent of the choice of the Cartan T and the choice of the fixed point. Thus it makes sense to talk about dominant characters of H.

To a character $\chi : H \to \mathbb{C}^*$ and a fixed point $x \in X^T$ we can associate a character $\chi : T \to \mathbb{C}^*$. We can then form

$$\mathrm{Ind}_{B_x}^G(\chi) \;=\; \{f : G \to \mathbb{C}^* \mid f(gb) = \chi(b^{-1})f(g) \;\text{ for all }\; b \in B_x\}.$$

If χ is dominant then these are the sections of the positive line bundle $\mathcal{L}_\chi$.

We now consider the following sheaf of functions on X, or, if you prefer, on $\tilde{X}$. For any $\tilde{x} \in \tilde{X}$ consider the map $r_{\tilde{x}} : H \to \tilde{X}$ given by $h \mapsto \tilde{x}h^{-1}$. Let us consider $\lambda \in \mathfrak{h}^*$. Let us write $\mathcal{O}_{H^{an}}(\lambda)$ for the subsheaf of $\mathcal{O}_{H^{an}}$ generated by the function $\exp(2\pi i\lambda)$. We set

$$\mathcal{O}_{X^{an}}(\lambda) = \{f \in \mathcal{O}_{\tilde{X}^{an}} \mid r_{\tilde{x}}^* f \in \mathcal{O}_{H^{an}}(\lambda) \;\text{ for all }\; \tilde{x} \in \tilde{X}\}\,.$$

We can now define the sheaf of differential operators $\mathcal{D}_{X^{an},\lambda}$ as $\mathrm{Diff}(\mathcal{O}_{X^{an}}(\lambda))$, the differential operators on $\mathcal{O}_{X^{an}}(\lambda)$.

Recall that we can describe $\mathcal{D}_{X^{an},\lambda}$ and hence $\mathcal{D}_{X,\lambda}$ algebraically as a subsheaf of $\tilde{\mathcal{D}}_X = (\pi_*\mathcal{D}_{\tilde{X}})^H$ by noting that $U(\mathfrak{h})$ is a central subsheaf of $\tilde{\mathcal{D}}_X$ which can then be specialized to λ or its infinitesimal neighborhoods. For λ regular dominant we have

$$\{U(\mathfrak{g})_{\widehat{\lambda}} - \text{modules}\} \cong \{\text{quasi-coherent } \mathcal{D}_{\widehat{\lambda}} - \text{modules}\}\,.$$

Furthermore, by Riemann-Hilbert, we have

$$P(X)_{-\widehat{\lambda}} \cong \{\text{regular holonomic } \mathcal{D}_{\widehat{\lambda}} - \text{modules}\}\,.$$

One can think of $P(X)_{\widehat{\lambda}}$ as $\mathfrak{h}$-equivariant perverse sheaves on $\tilde{X}$ where the isotropy group $\pi_1(H) = \mathfrak{h}_{\mathbb{Z}} = \operatorname{Ker}(\exp : \mathfrak{h} \to H)$ acts via generalized eigenvalue $\exp(2\pi i\lambda)$. In other words a perverse sheaf $\mathcal{F}$ on $\tilde{X}$ lies in $P(X)_{\widehat{\lambda}}$ if for all $\tilde{x} \in \tilde{X}$ the $r_{\tilde{x}}^*\mathcal{F}$ is a local system with generalized eigenvalue $\exp(2\pi i\lambda)$.

2. Real Groups

Consider an anti-involution $\sigma : G \to G$ such that $G_{\mathbb{R}} = G^{\sigma}$ is a real group. If we choose a maximal compact subgroup $K_{\mathbb{R}} \subset G_{\mathbb{R}}$, then there exists $\theta : G_{\mathbb{R}} \to G_{\mathbb{R}}$ such that $G_{\mathbb{R}}^{\theta} = K_{\mathbb{R}}$. By complexifying we get $\theta : G \to G$ and then $K = G^{\theta}$ is the complexification of $K_{\mathbb{R}}$.

This way we get

$$\{\text{Conjugate linear involutions}\} \longleftrightarrow \{\text{Involutions}\},\ \sigma \leftrightarrow \theta.$$

Note that $G_{\mathbb{R}}$ and K do not need to be connected.

Example 2.1.

$$\begin{aligned} G &= SL(2,\mathbb{C}) & G &= PGL_2(\mathbb{C}) \\ G_{\mathbb{R}} &= SL(2,\mathbb{R}) & G_{\mathbb{R}} &= PGL_2(\mathbb{R}) \\ K &= SO(2,\mathbb{C}) & K &= O(2,\mathbb{C}). \end{aligned}$$

The $O(2,\mathbb{C})$ and $PGL_2(\mathbb{R})$ are disconnected.

We consider representations of $G_{\mathbb{R}}$ on complete, locally convex, Hausdorff topological complex vector spaces,

$$\pi : G_{\mathbb{R}} \to \operatorname{Aut}_{\mathbb{C}}(V)$$

such that $G_{\mathbb{R}} \times V \to V$ is continuous.

We consider a subclass of representations defined by Harish-Chandra which includes all irreducible unitary representations.

We have

$$U(\mathfrak{g}) \subset \mathcal{Z}(\mathfrak{g}) = U(\mathfrak{h})^W,\ \mathfrak{g} = \operatorname{Lie} G$$

where $\mathfrak{h}$ is the universal Cartan and the action of W is the dot-action. Recall that the universal Cartan has a canonical set of positive roots. Given $\lambda \in \mathfrak{h}^*$, we have

$$\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}.$$

We consider representation (π, V) of $G_{\mathbb{R}}$ such that

$$M = V_{K_{\mathbb{R}}} := \{K_{\mathbb{R}} \text{ finite vectors}\}$$

has all irreducible representations appearing with finite multiplicity. These are the admissible representations.

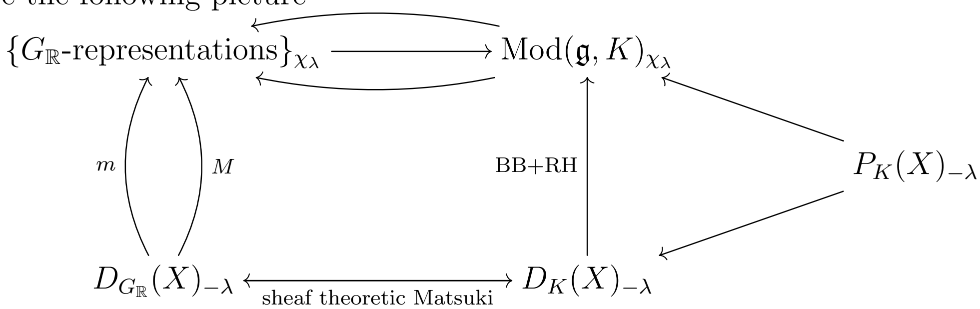
Now $\mathfrak{g}$ acts on M, want M to be finitely generated over $U(\mathfrak{g})$ and we assume that $\mathcal{Z}(\mathfrak{g})$ acts on M via a character χ_λ for some $\lambda \in \mathfrak{h}^*$. Then M is a $(\mathfrak{g}, K)$-module, Harish-Chandra module, with infinitesimal character χ_λ (or λ).

We write $\mathrm{Mod}(\mathfrak{g},K)_\lambda$ for the category of $(\mathfrak{g},K)$-modules. We could also consider $\mathrm{Mod}(\mathfrak{g},K)_{\hat{\lambda}}$, these M are annihilated by some power of $z-\chi_\lambda(z)$. Note that we could also consider pro-objects.

$$\{G_{\mathbb{R}}\text{-representations}\}_{\chi_\lambda} \longrightarrow \mathrm{Mod}(\mathfrak{g},K)_{\chi_\lambda}$$

The forgetful functor $V \mapsto V_{K_{\mathbb{R}}}$ has both adjoints, the minimal realization (real analytic functions) and the maximal realization (hyperfunctions).

We have the following picture

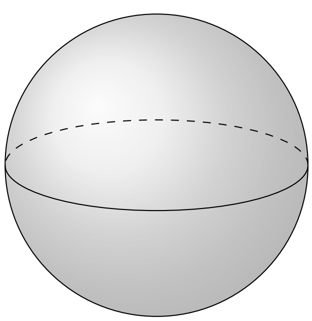


The functor $P_K(X)_{-\lambda} \to \mathrm{Mod}(\mathfrak{g},K)_{\chi_\lambda}$ is an equivalence if λ is regular dominant. If λ is dominant but not regular then we obtain $\mathrm{Mod}(\mathfrak{g},K)_{\chi_\lambda}$ from $P_K(X)_{-\lambda}$ by localizing at a Serre subcategory.

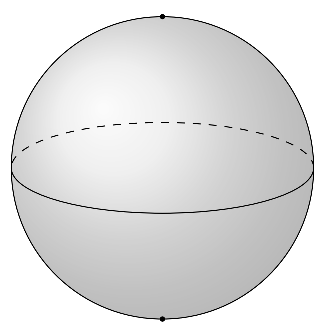
We have

$$m(\mathcal{F}) = H^*(\mathcal{B}, \mathcal{F} \otimes \mathcal{O}^{\mathrm{an}}(\lambda))$$
$$M(\mathcal{F}) = \mathrm{Ext}^*(\mathbb{D}\mathcal{F}, \mathcal{O}^{\mathrm{an}}(\lambda))$$

Example 2.2. $SL(2,\mathbb{R})$



$G_{\mathbb{R}}$



$(\mathfrak{g},K)$

Closer to representations

duality is Verdier duality

character formulas

Standard algebraic geometry

$$\{G_{\mathbb{R}}\text{-orbits}\} \longleftrightarrow \{K\text{-orbits}\}$$

If the group G is complex to begin with then both incarnations give us $D_G(X\times X)_{-\widehat{\lambda}}$. Ignoring one copy of the center of $U(\mathfrak{g}\times\mathfrak{g})$ this amounts to $D_B(X)_{-\widehat{\lambda}}$, i.e., we are in the context of category $\mathcal{O}$.

Excercise 2.1. *Note that the functors m and M provide a direct construction of representations from topological data bypassing $\mathcal{D}$-modules. You can use the same construction for $\mathcal{F} \in D_K(X)_{-\lambda}$. When you do that you get something very big. Show that if you pass to K-finite vectors you get the same result as you would by inverting the RH-correspondence and then taking global sections.*

3. Category $\mathcal{O}_{\widehat{0}}$ and $\mathcal{O}_0'$ for $\mathfrak{sl}(2)$.

Let us recall that, for any Lie algebra $\mathfrak{g}$ the category $\mathcal{O}_{\widehat{0}}$ is equivalent to $P_B(\widetilde{X})$ and similarly $\mathcal{O}_0'$ is equivalent to $P_N(X)$.

For $\mathfrak{sl}(2)$ the flag manifold is $X = \mathbb{CP}^1$ and $\tilde{X} = \mathbb{C}^2 - \{0\}$. The H-fibration is the usual map $\pi : \mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$ where $\mathbb{C}^*$ acts on $\tilde{X}$ via the inverse. Let us think of the $\mathbb{C}^2 - \{0\}$ in coordinates (x, y) with the usual action of $SL(2)$. Let us consider the concrete Cartan $T = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. It fixes two Borels $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right)$ which correspond to points $[1, 0]$ and $[0, 1]$, respectively., i.e., $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ fixes $[1, 0]$ and $\left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right)$ fixes $[0, 1]$. If we write $s \in \mathbb{C}^* = H$ then the identification by the point $[1, 0]$ sends $s \mapsto \left(\begin{smallmatrix} s & 0 \\ 0 & s^{-1} \end{smallmatrix}\right)$ and the identification by the point $[0, 1]$ sends $s \mapsto \left(\begin{smallmatrix} s^{-1} & 0 \\ 0 & s \end{smallmatrix}\right)$. In both cases the positive root is given by $s \mapsto s^{-2}$. Thus, -2 is the positive root in $\mathfrak{h}$

As a reality check, let us recall the maps $r_{\widetilde{x}} : H \to \widetilde{X}$ given by $h \mapsto \widetilde{x} h^{-1}$. In our coordinates we get maps $\mathbb{C}^* \to \mathbb{C}^2 - \{0\}$ given by $s \mapsto (s^{-1}x, s^{-1}y)$. The finite dimensional representations of $SL(2)$ are given by homogenous polynomials in x, y which thus correspond to negative powers of s.

Let us now consider the Borel $B = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ and use it and its unipotent radical N to consider the categories $P_B(\widetilde{X})$ and $P_N(X)$. For $\mathfrak{sl}(2)$ it is easy to see that by considering the -2 and 0 weight spaces both the categories $\mathcal{O}_{\widehat{0}}$ and $\mathcal{O}_0'$ are equivalent to quivers

$$A \underset{p}{\overset{q}{\leftrightarrows}} B \qquad \text{with} \qquad q \circ p = 0\,;$$

here A is the 0-weight space, B is the -2-weight space, q is given by e and p is given by f. In the world of perverse sheaves and $\mathcal{D}$-modules A is the generic stalk and B is the microlocal stalk, but this would lead us to a separate discussion.

As can be immediately deduced from the quiver description, the categories $\mathcal{O}_{\widehat{0}}$ and $\mathcal{O}_0'$ have five indecomposable objects four of which are just standard, co-standard, and irreducible modules which are easy to describe in any incarnation. We will proceed to describe the projective P_{-2} in both settings.

Let us first consider $\mathcal{O}_0'$. The N-orbits on X consist of the closed orbit $\infty = [1, 0]$ and the rest. If we write z for the coordinate y/x on the coordinate batch $x \neq 0$ then on that batch the projective is given by the $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}z\partial_z z$.

To write down the corresponding perverse sheaf we proceed as follows. Let us consider $U = \mathbb{CP}^1 - \{\infty, c\}$. Let us now consider the family of perverse sheaves P_c on $\mathbb{CP}^1$ by extending the constant sheaf (in degree -1) from U to $\mathbb{CP}^1$ by taking the !-extension across

∞ and the $*$-extension (in the derived sense) across the point c. The we have

$$P_{-2} = \lim_{c\to\infty} P_c\,. \tag{3.1}$$

The limit should be thought of taken in the sense of nearby cycles. One can thus view P_{-2} as an extension of the constant sheaf where we simultaneously extend by ! and $*$ across ∞.

Note that P_{-2} is not B-equivariant and hence it does not lie in $\mathcal{O}_{\widehat{0}}$. This is more or less clear by observing that the construction above is not $T = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right)$-equivariant.

Thus, the projective P_{-2} of $\mathcal{O}_{\widehat{0}}$ does not live on X. However, we can construct it on $\widetilde{X}$ using the same idea. Note that B has two orbits on $\widetilde{X}$. One of them is the x-axis and the complement is isomorphic to B. Let us now consider the following subset

$$U \;=\; \mathbb{C}^2 - (\{(ca, a^{-1}) \in \mathbb{C}^2 - \{0\} \mid a \in \mathbb{C}^*\} \cup \{(x,0) \in \mathbb{C}^2 - \{0\} \mid x \in \mathbb{C}\})\,.$$

We start with a constant sheaf on U in degree -2 and then take a !-extension across the x-axis and the $*$-extension across $\{(ca, a^{-1}) \in \mathbb{C}^2 - \{0\} \mid a \in \mathbb{C}^*\}$ and call the resulting sheaf P_c. Note that for any c this sheaf is T-equivariant. We will now set as before.

$$P_{-2} = \lim_{c\to\infty} P_c\,. \tag{3.2}$$

Now P_{-2} is T-equivariant by construction. It is also N-equivariant as it is constructible along the B-orbits. Hence, it is B-equivariant. Note, however, that it is not H-equivariant and hence is not a sheaf on X.