Beilinson-Bernstein localisation again

Recall:

Thm: Let G be a semisimple algebraic group w/ Lie algebra $\mathfrak{g}$. If $\lambda \in \mathfrak{h}^*$ is a regular dominant weight, then there's an equivalence of categories

$$\left\{ \mathcal{D}^{\lambda}_{G/B}\text{-modules} \right\} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \left\{ U^{\lambda}(\mathfrak{g})\text{-modules} \right\}$$

where $U^{\lambda}(\mathfrak{g}) = U(\mathfrak{g}) / \{ x - \lambda(x) \mid x \in \underset{\substack{\shortparallel \\ Z(U(\mathfrak{g}))}}{Z} \}$.

Goal for today:

1) Explain what $\mathcal{D}^{\lambda}_{G/B}$ means for general $\lambda \in \mathfrak{h}^*$
2) Introduce "monodromic $\mathcal{D}$-modules"
3) Examples!

Main reference:

Beilinson and Bernstein, "A proof of the Jantzen conjectures".

§1: Twisted differential operators and monodromic D-modules

Let X be a smooth alg variety / $\mathbb{C}$

Def^n: An algebra of twisted differential operators (or tdo) on X is a sheaf of associative algebras D on X equipped with a morphism of algebras $i: \mathcal{O}_X \to D$ such that there exists a filtration $D_\bullet$ on D with

1) $D_\bullet$ is a ring filtration with $D_{-1} = 0$ and $gr_\bullet D$ commutative

2) $D_0 = \mathcal{O}_X$ and
$$S_{\mathcal{O}_X}(D_1/D_0) \longrightarrow gr_\bullet D \quad \text{is an iso}$$

3) The morphism $\sigma : D_1/D_0 \to T_X$ given by
$$\sigma(\partial)(f) = \partial f - f\partial \in D_0 = \mathcal{O}_X$$
is an isomorphism.

In modern language: D is a filtered quantisation of the cotangent bundle T^*X.

Another perspective: Lie algebroids

Def^n: A Lie algebroid on X is a quasi-coherent sheaf $\mathcal{F}$ on X equipped with
1) A Lie bracket $[\,,\,]: \mathcal{F} \otimes_{\mathbb{C}} \mathcal{F} \to \mathcal{F}$ (not $\mathcal{O}$-linear!)
2) An anchor map $a: \mathcal{F} \to T_X$

such that
$$[u, fv] = a(u)(f)v + f[u,v]$$
for u, v local sections of $\mathcal{F}$, f local section of $\mathcal{O}_X$.

Eg. the T_X is a Lie algebroid.
Eg. any Lie alg is a Lie algebroid

If $\mathcal{D}$ is a tdo, then $\widetilde{T} = \mathcal{D}_1$ is a Lie algebroid equipped w, an exact sequence
$$0 \to \mathcal{O}_X \xrightarrow{i} \widetilde{T} \xrightarrow{a} T_X \to 0$$
such that $1 \in \mathcal{O}_X$ is central.
Such a thing is called a Picard algebroid. The functor
$$\{\text{tdos}\} \longrightarrow \left\{\begin{array}{c}\text{Picard} \\ \text{algebroids}\end{array}\right\}$$
$$\mathcal{D} \longmapsto \mathcal{D}_1$$
is an equivalence of categories.

Inverse sends $\tilde{T}$ to $\tilde{D} = U(\tilde{T}) / \langle i(f) - f \mid f \in \mathcal{O}_X \rangle$

universal enveloping algebra

<u>Monodromic D-modules</u>: (source of examples of tdos)

Let H be a torus, $\pi: \tilde{X} \to X$ a principal H-bundle.

A <u>monodromic D-module</u> on X is an H-equivariant $D_{\tilde{X}}$-module.

Let $\tilde{D} = (\pi_* D_{\tilde{X}})^H$. Then we have

$$\left\{ \begin{matrix} \text{Monodromic} \\ D\text{-modules} \end{matrix} \right\} \simeq \left\{ \begin{matrix} \tilde{D}\text{-modules} \\ \text{on } X \end{matrix} \right\}$$

We have $\tilde{D} = U(\tilde{T})$ where

$\tilde{T} = (\pi_* T_{\tilde{X}})^H$ (Lie algebroid)

Have extension

$$0 \longrightarrow \mathfrak{h} \otimes \mathcal{O}_X \longrightarrow \tilde{T} \longrightarrow T_X \longrightarrow 0$$

fibre directions $\quad \mathfrak{h} = \mathrm{Lie}(H)$

with $\mathfrak{h}$ central

$\Rightarrow \tilde{D}$ is an algebra over $S(\mathfrak{h})$

Easy fact: For $\lambda \in \mathfrak{h}^*$, $D^\lambda := \tilde{D} \otimes_{S(\mathfrak{h})} \mathbb{C}_\lambda$ is a tdo.

E.g. $X = \mathbb{P}^1$, $\tilde{X} = \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$, $H = \mathbb{G}_m$.

$$\left\{ \begin{array}{l} \text{Monodromic} \\ D\text{-modules} \\ \text{on } \mathbb{P}^1 \end{array} \right\} = \frac{\text{Graded } \mathbb{C}[x, y, \partial_x, \partial_y]\text{-modules}}{\{\text{modules supported at } x = y = 0\}}$$

Use $\mathrm{QCoh}(\mathbb{C}^2 \setminus \{0\}) = \dfrac{\mathrm{QCoh}(\mathbb{C}^2)}{\{\text{mods supported at } x = y = 0\}}$

$S(\mathfrak{h}) = \mathbb{C}[\tilde{h}]$ acts by $\tilde{h} = x\partial_x + y\partial_y$.

<u>Charts</u>: $U = \{y \neq 0\} = \mathrm{Spec}\, \mathbb{C}[z]$, $z = \frac{x}{y}$.

$V = \{x \neq 0\} = \mathrm{Spec}\, \mathbb{C}[w]$, $w = \frac{y}{x}$.

$$\Gamma(U, \tilde{D}) = \text{Deg } 0 \text{ part of } \mathbb{C}[x, y, y^{-1}, \partial_x, \partial_y]$$

$$= \frac{\mathbb{C}\langle z = \frac{x}{y}, \tilde{h} = x\partial_x + y\partial_y, y\partial_x \rangle}{[\tilde{h}, z] = 0, \ [y\partial_x, z] = 1, \ [\tilde{h}, y\partial_x] = 0}$$

$$\cong \mathbb{C}[z, \tilde{h}, \partial_z], \quad \partial_z = y\partial_x.$$

$$\Gamma(V, \tilde{D}) = \frac{\mathbb{C}\langle w = \frac{y}{x},\ \tilde{h} = x\partial_x + y\partial_y,\ x\partial_y \rangle}{[\tilde{h}, w] = 0,\ [\tilde{h}, x\partial_y] = 0,\ [x\partial_y, w] = 1}$$

$$\cong \mathbb{C}[w, \tilde{h}, \partial_w] \qquad \partial_w = x\partial_y$$

Change of coordinates:

$$w \longmapsto z^{-1}$$

$$\partial_w \longmapsto x\partial_y = \frac{x}{y}(y\partial_y + x\partial_x) - \frac{x^2}{y^2} y\partial_x$$

$$= \tilde{h}z - z^2\partial_z .$$

D^λ : set $\tilde{h} = \lambda$ in above.

§2: Monodromic D-modules on flag varieties

Let G be a semisimple alg grp $/\mathbb{C}$

$\supset$

$B = B^-$ Borel subgroup (negative roots)

$\supset$

N^- unipotent radical of B^-

$H = B^-/N^-$ Cartan subgroup ($\cong$ max torus)

G/B^- flag variety

Have commuting actions

$$G \circlearrowright G/N^- \circlearrowleft H \qquad \begin{array}{l} g\cdot(xN^-) = gxN^- \\ h\cdot(xN^-) = xhN^- \end{array}$$

$$\begin{array}{c} G \circlearrowright G/N^- \circlearrowleft H \\ \pi \big\downarrow \quad \longleftarrow \text{Principal } H\text{-bundle.} \\ G \circlearrowright G/B^- \end{array}$$

$\Rightarrow$ Monodromic D-modules setup applies.

Get $\tilde{D}$ sheaf of assoc. algebras on G/B with $S(\mathfrak{h}) \subseteq$ centre, and

$$D^\lambda = \tilde{D} \otimes_{S(\mathfrak{h})} \mathbb{C}_\lambda.$$

The $G\times H$-action gives a homomorphism

$$U(\mathfrak{g}) \otimes S(\mathfrak{h}) \longrightarrow \Gamma(G/B, \tilde{D}) = \Gamma(G/N, D_{G/N})^H$$

<u>Fact</u>: This factors through an isomomorphism

$$U(\mathfrak{g}) \otimes_Z S(\mathfrak{h}) \xrightarrow{\sim} \Gamma(G/B, \tilde{D})$$

where $Z = Z(U(\mathfrak{g})) \cong S(\mathfrak{h})^W$ ← $\circ$-action $w\circ\lambda = w(\lambda+\rho)-\rho$ on $\mathfrak{h}^*$.

↑ Harish-Chandra iso

(defined so that $S(\mathfrak{h})$ and Z actions on highest weight vectors agree)

Example: $G = SL_2$. $B = B^- = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$.

$G/N^- \xrightarrow{\sim} \mathbb{C}^2 \setminus \{0\}$ $\quad N^- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} N^- \longmapsto \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\tilde{D} = \mathbb{C}[x, y, \partial_x, \partial_y]$ as in previous example

$\mathfrak{g}$-action?

$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$

So (right) action on functions is $e \cdot x = y$
$e \cdot y = 0$

This is $[\cdot, -y\partial_x]$

$\rightsquigarrow$ map e to $-y\partial_x$

Similarly, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto -x\partial_y$

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto y\partial_y - x\partial_x$

For H-action, similar calculation gives

$\tilde{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h} \longmapsto x\partial_x + y\partial_y$.

distinguish from $h \in \mathfrak{g}$

I is generated by the Casimir

$ef + fe + \frac{1}{2}h^2$

Check that

$$
\begin{aligned}
ef + 2fe + h^2 &= y\partial_x x\partial_y + x\partial_y y\partial_x \\
&\quad + \tfrac{1}{2}(y\partial_y - x\partial_x)^2 \\
&= \tfrac{1}{2}(x\partial_x + y\partial_y)^2 + (x\partial_x + y\partial_y) \\
&= \tilde{h}^2 + \tfrac{\tilde{h}}{2} \\
&= \tfrac{1}{2}\tilde{h}(\tilde{h}+2)
\end{aligned}
$$

w_0-action sends $\tilde{h}$ to $-\tilde{h}-2$.

Summary: We have:

- $\tilde{D}$ sheaf of $S(\mathfrak{h})$-algs on G/B.
- $D^\lambda = \tilde{D} \otimes_{S(\mathfrak{h})} \mathbb{C}_\lambda$ twisted diff ops

$$\Gamma(\tilde{D}) = U(\mathfrak{g}) \otimes_{Z} S(\mathfrak{h})$$

$$\Gamma(D^\lambda) = U(\mathfrak{g}) \otimes_{Z} \mathbb{C}_\lambda = U^\lambda(\mathfrak{g})$$

Beilinson-Bernstein:

Assume $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for α positive root.

Then

$$\Gamma : D^\lambda\text{-mod} \longrightarrow U^\lambda(\mathfrak{g})\text{-mod}$$

is an equivalence.

Rest of talk: Your favourite $U(\mathfrak{g})$-mod as a D^λ-module.

Main examples:

Rem: defined to be N^+-integrable.

Verma modules $M(\lambda)$ (and $M(w\cdot\lambda)$)

Dual Vermas $M^*(\lambda)$ (and $M^*(w\cdot\lambda)$)

Irreducible modules $L(\lambda)$ (and $L(w\cdot\lambda)$)

E.g. $G = SL_2$:

$$M(\lambda) = \frac{U^\lambda(\mathfrak{g})}{U^\lambda(\mathfrak{g})\text{-span}\{e, h-\lambda\}}$$

Must map to

$$\frac{D^\lambda}{D^\lambda\text{-span}\{e, h-\lambda\}} =: M$$

On charts: $z = \frac{x}{y}$, $e = -\partial_z$

$h = \lambda - 2z\partial_z$

$\rightsquigarrow$ D-mod is $\mathcal{O}$ with $\partial_z(1) = 0$

$f = z^2\partial_z - \lambda z$

$w = \frac{y}{x}$, $e = w^2\partial_w - \lambda w$

$f = -\partial_w$

$\rightsquigarrow$ D-mod is $\frac{D}{D(w\partial_w - \lambda)}$

$h = 2w\partial_w - \lambda$

$M(w\cdot\lambda) = M(-\lambda-2)$?

$$\frac{D^\lambda}{D^\lambda\text{-span}\{e, h+\lambda+2\}}$$

<u>On charts</u>: • z chart. Kill ∂_z and $2\lambda+2$

$\Rightarrow$ module is zero since $\lambda \neq -1$.

• w chart.

Kill $w^2\partial_w - \lambda w$ and $w\partial_w + 1$

$\Leftrightarrow$ Kill $w \cdot$ generator

$\leadsto$ D-module is $\dfrac{\mathbb{C}[w,\partial_w]}{\mathbb{C}[w,\partial_w]w} \cong \mathbb{C}[\partial_w]$

(supported at 0).

<u>Dual Vermas</u>:

Apply "Verdier" duality to D-mod for $M(w\cdot\lambda)$

For $M(-\lambda-2)$: No change

For $M(\lambda)$: $j_*\mathcal{O}$, j: (z chart) $\hookrightarrow \mathbb{P}^1$

i.e. functions with poles at $z=\infty$

D-action in w chart is $\partial_w f = f'(w) + \frac{\lambda}{w} f(w)$.

(<u>Note</u>: This is only correct Verdier dual if $\lambda \notin \mathbb{Z}_{\leq 0}$.)

Irreducible: There's a map

$M(\lambda) \longrightarrow M^*(\lambda)$ sending 1 to 1.

$L(\lambda)$ = image.

In general:

Have $\mathring{X}_w = N^+ w B^- / B^- \xhookrightarrow{j_w} G/B^-$

H-bundle has unique N^+-equivariant trivialisation here

$\Rightarrow$ D^λ-mods on $\mathring{X}_w$ $\leftrightarrow$ D-mods on $\mathring{X}_w$

$M(w\cdot\lambda) \leftrightarrow j_{w!}(\mathcal{O}_{\mathring{X}_w})$ unique N^+-invariant flat connection

$M^*(w\cdot\lambda) \leftrightarrow j_{w*}(\mathcal{O}_{\mathring{X}_w})$

$L(\lambda) \leftrightarrow \mathrm{im}\left(j_{w!}(\mathcal{O}_{\mathring{X}_w}) \longrightarrow j_{w*}(\mathcal{O}_{\mathring{X}_w})\right)$