Convergence analysis of feedback-based iterative learning control with input saturation ⋆

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Abstract

The use of feedback-based iterative learning control (ILC) has been widely reported in the literature to reduce large transient errors in the iteration-domain. Using both feedback and feed-forward (ILC) would result in large control input. However, due to the existence of hardware limitation, input constraints exist. Such a hard constraint might lead to unacceptable performances such as unstable trajectories in the time-domain and steady-state error in the iteration-domain. In this paper, a feedback-based ILC design is proposed for a class of linear-time-invariant system in the presence of input constraints. The proposed control scheme consists of two parts: one (feedback) deals with the performance in time domain while the other (ILC) ensures ensure the perfect tracking performance. With the help of composite energy function, it is shown that the proposed algorithm can ensure the perfect tracking performance in the presence of hard input constraints under appropriate assumptions. Simulation results support the theoretical findings.

Key words: iterative learning control, input saturation, composite energy function

1 Introduction

The field of iterative learning control (ILC) stems from the seminal work of [4] in trajectory tracking of robotic systems. By "learning" from previous iterations, an ILC algorithm can achieve perfect tracking performance without the precise knowledge of engineering systems. It finds applications in many engineering systems where the system performs the same task over iterations with a high tracking precision requirement. Examples include chemical batch processes [16,17], robotic manufacturing systems [13,15], precision motion control systems [1,5], traffic flow control [10], control of wind turbine rotors [27], modelling of human motor learning [32], and robotic rehabilitation [8,31]. More applications can be found in the survey papers [6,18] and references therein.

The idea behind the ILC is "practice makes perfect". The errors observed in previous iterations can be used to correct the behaviour of the current iteration. Thus, ILC algorithms have a feed-forward nature. As past information is used in updating the current iteration, the requirement of model knowledge is relaxed. Thus the ILC scheme is described as a "model-free" or "data-driven" control method.

Two major analysis tools have been used in the design of ILC for continuous-time systems. One is the contraction mapping (CM) method and the other is composite energy function (CEF) [28]. Both approaches focus on convergence in the iteration-domain by ignoring the dynamics in the time-domain. Two methods can also be unified as indicated in [25].

A pure feed-forward ILC is relatively simple to design using CM methods. However, without the feedback, the tracking performance of first few iterations may not be acceptable as in many applications, the output needs to be constrained in some safe regions. For example, in chemical processes or rehabilitation robotic systems, the output signals usually have hard constraints as well. In the sequel, feedback is introduced to work together with ILC to reduce the tracking error in the transient of iteration-domain as well as to improve the robustness to non-repetitive disturbances [9,30].

In this paper, the feedback-based ILC refers to the control structure where the control input has the feedback component to ensure the boundedness of the trajectories while the ILC learns the desired control input. The feedback can be either state feedback or output feed-

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back. Output feedback works well for output tracking problems when the internal state is well behaved. Under such a situation, for a linear-time-invariant (LTI) system, frequency domain analysis using CM is sufficient to ensure the convergence within the frequency band of interests [3,7,12]. When the system is non-minimum phase or nonlinear, the output feedback might not be sufficient to ensure the boundedness of the state. In particular, when the system is nonlinear, finite escape of the state might happen without affecting the convergence of the output [14]. Hence this work considers a feedback-based ILC scheme that has both the state feedback and an ILC to track the desired output.
The input constraints exist due to actuator limitations. It is highlighted that an ILC algorithm is typically an integrator over the iteration-domain, *i.e.* the update law generally takes the following form:

$$u_{i+1}(t) = u_i(t)+f_u(e_{i+1}(t), e_i(t), \cdots, e_{i+1-k}),\ t \in [0,\ T_f],$$

where subscript i represents the iteration number, u is the control input and k is some integer. Therefore, it is possible to generate an unbounded control input as the iteration number goes to infinity. Due to input hard constraints, such an unbounded input is not feasible. For continuous-time systems, input constraints have been addressed in for ILC in [25,29] and [24]. In these schemes, the current feedback is used to track the desired state trajectory using CEF method.

To the best of the authors' knowledge, input saturation has not to be considered for feedback-based ILC schemes, in which the feedback design and feed-forward ILC are decoupled. This paper differs from the majority of feedback-based ILC design where state-feedback is used instead of output feedback to ensure that all the state signals are well-behaved. The role of feed-forward (ILC) is to ensure convergence by learning. The contributions of this paper are highlighted as follows:

(1) A new updating structure is proposed to handle input saturation for feedback-based ILC. It is highlighted that the feedback-based ILC is first designed without taking input saturation into consideration. In order to handle the actuator saturation, the feed-forward ILC is modified. By fully exploiting the properties of the saturation function, the proposed structure can ensure a perfect tracking performance when the desired output trajectory is feasible within the saturation bound.
(2) The feed-forward ILC is designed based on the CM method with a convergence condition. However a novel CEF technique is used to analyse the convergence in the presence of input saturation. The CEF is composed of tracking error in terms of the state of the past trial as well as the $\mathcal{L}^2$ norm error between feed-forward control input and the desired control input at the current trial. When the convergence condition is satisfied, the proposed control algorithm can ensure the convergence of the tracking error with input saturation with the help of this novel CEF as shown in the main result (Theorem 1).

It should be noted that to demonstrate our idea to handle input saturation for feedback-based ILC, a simple multi-input-multi-output (MIMO) LTI system with a relative degree of one is used in this paper. However, the analysis can be extended to general nonlinear time-varying systems with a higher relative degree by using a similar CEF.
The structure of the paper is organised as follows. Section 2 provides the preliminaries and presents the problem formulation. The proposed algorithm is presented in Section 3, followed by an analysis of boundedness of trajectories in the presence of input saturation. In Section 4, the convergence of the proposed control algorithm is investigated using a novel CEF followed by an illustrative example in Section 5. Section 6 concludes the paper.

2 Preliminaries and Problem Formulation

We use the following notations in this paper. $\mathcal{R}$ and $\mathcal{N}$ represents the set of real and natural numbers respectively. $\mathcal{R}_+$ denotes the set of positive real numbers. For any vector $\mathbf{x} \in \mathcal{R}^n$ represents a vector with n elements, i.e $\mathbf{x} = [x^1,\ x^2, \cdots,\ x^n]^\intercal$ and $|\mathbf{x}|$ denotes the Euclidean norm where $|\mathbf{x}| = \sqrt{\mathbf{x}^\intercal \mathbf{x}}$. For any matrix $A \in \mathcal{R}^{n\times n}$, $|A|$ represents the induced matrix norm. $A = A^T > 0$ indicates A is a symmetric positive definite matrix. For a symmetric positive definite matrix A, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ represent its smallest and its largest eigenvalue respectively. The matrix I_n denotes the identity matrix of dimension n. The symbol $i \in \mathcal{N}$ denotes the iteration number and the subscript $(\cdot)_i$ indicates the signal at the i^{th} iteration. For any $j \in \mathcal{N}$, the set of all continuous function in $[0,\ T_f]$ that is differentiable upto j^{th} order is represented by $\mathcal{C}^j[0,\ T_f]$. Next, a few definitions are introduced.

Definition 1 *For any* $\mathbf{x}(\cdot) \in \mathcal{C}[0,\ T_f]$, *the supremum norm is defined as* $\|\mathbf{x}\|_s \triangleq \max_{t\in[0,\ T]} |\mathbf{x}(t)|_\infty$, *and the* λ *- norm is defined for a given* $\lambda > 0$ *as* $\|\mathbf{x}\|_\lambda \triangleq \max_{t\in[0,\ T_f]} e^{-\lambda t}\,|\mathbf{x}(t)|_\infty$, *where* $|\mathbf{x}|_\infty = \max_{j\in[1,\ 2,\ \cdots,\ n]} |x^j|$.

Remark 1 *The supremum norm and the* λ *- norm are equivalent [28, Chapter 2]. These two norms can be used to show the uniform convergence of the tracking error in the finite time interval* $[0,\ T_f]$. ○

Definition 2 *For any* $\mathbf{x} \in \mathcal{C}[0,\ T_f]$ *the* $\mathcal{L}^2$ *norm is defined as* $\|\mathbf{x}\|_{\mathcal{L}^2} \triangleq \left(\int_0^{T_f} |\mathbf{x}(\tau)|^2 d\tau\right)^{\frac{1}{2}}$ *where as the* $\mathcal{L}_e^2$ *norm* *is defined as* $\|\mathbf{x}\|_{\mathcal{L}_e^2} \triangleq \left(\int_0^{T_f} e^{-\lambda\tau} |\mathbf{x}(\tau)|^2 d\tau\right)^{\frac{1}{2}}$, *for any* $\lambda > 0$.

Remark 2 *For any* $\mathbf{x} \in \mathcal{C}[0,\ T_f]$, *its* $\mathcal{L}^2$ *norm and* $\mathcal{L}_e^2$ *norm are equivalent. So the convergence in terms of* $\mathcal{L}_e^2$ *norm indicates the convergence of* $\mathcal{L}^2$. ◦

Definition 3 *The saturation function for any* $u \in \mathcal{R}$ *is defined as* $\mathrm{sat}(u, u^*) \triangleq sign\,(u)\, min\{u^*, |u|\}$ *where* $u^* > 0$ *is a scalar constant. For any* $\mathbf{u} \in \mathcal{R}^m$, $\mathrm{sat}(\mathbf{u}, \mathbf{u}^*) = \left[\mathrm{sat}(u^1, u^{1^*}), \cdots, \mathrm{sat}(u^m, u^{m^*})\right]^\intercal$ *where* $\mathbf{u}^* = [u^{1^*}, \cdots, u^{m^*}]$ *is a vector of constants with* $u^{j^*} > 0, \forall\ j \in [1,\ 2,\ \cdots,\ m]$.

In order to facilitate the proof of main theorem, the following lemmas are used.

Lemma 1 *For any given* $\mathbf{u}_r$, *and* $\mathbf{u} \in \mathcal{R}^m$ *and* $\mathbf{u}^* \in \mathcal{R}_+^m$ *satisfying* $\mathrm{sat}(\mathbf{u}_r, \mathbf{u}^*) = \mathbf{u}_r$ *and if* $\delta\mathbf{u}$ *and* $\delta\widetilde{\mathbf{u}}$ *are defined as* $\delta\mathbf{u} \triangleq \mathbf{u}_r - \mathbf{u}$ *and* $\delta\widetilde{\mathbf{u}} \triangleq \mathbf{u}_r - \mathrm{sat}(\mathbf{u}, \mathbf{u}^*)$ *respectively, then* $|\delta\widetilde{\mathbf{u}}|^2 \le |\delta\mathbf{u}|^2$ *[29, Property-3].* □

Lemma 2 *Let* $\mathbf{v},\ \mathbf{u},\ \mathbf{w} \in \mathcal{R}^m$ *and* $\mathbf{u}^* \in \mathcal{R}_+^m$. *If* $\mathbf{v} = \mathrm{sat}(\mathbf{u}, \mathbf{u}^*) + \mathbf{w}$, *then* $|\mathrm{sat}(\mathbf{v}, \mathbf{u}^*) - \mathbf{v}| \le |\mathbf{w}|$ *holds true [29, Property-4].* □

2.1 Plant Model

For the simplicity of presentation, consider an LTI MIMO square system [1] represented in the state space as

$$\begin{aligned}\dot{\mathbf{x}}_i(t) =& A\mathbf{x}_i(t) + B\mathbf{u}_i(t)\\ \mathbf{y}_i(t) =& C\mathbf{x}_i(t),\end{aligned} \tag{1}$$

where $\mathbf{x}_i \in \mathcal{R}^n$ and $\mathbf{y}_i, \mathbf{u}_i \in \mathcal{R}^m$ are the state, output and input at the i^{th} iteration respectively and matrices (A, B, C) have appropriate dimensions.

Assumption 1 *For system* (1), (A, B) *is controllable.* □

The controllability of system (1) indicates the design freedom in placing the system closed loop eigenvalues arbitrarily. For simplicity, we assume that the state is measurable. It is noted that in the design of the feedforward ILC, the matrices (A, B, C) are not completely known. The only information needed is the convergence condition (see Equation (8)).

Assumption 2 *For a given reference trajectory* $\mathbf{y}_r \in \mathcal{C}^1[0,\ T_f]$ *and a given input saturation limit* $\mathbf{u}^* \in \mathcal{R}_+^m$, *there exists a reference input* $\mathbf{u}_r \in \mathcal{C}[0,\ T_f]$, *and a reference state* $\mathbf{x}_r \in \mathcal{C}^1[0,\ T_f]$ *that satisfy the system dynamics* (1), *such that*

$$\begin{aligned}\dot{\mathbf{x}}_r(t) =& A\mathbf{x}_r(t) + B\mathbf{u}_r(t)\\ \mathbf{y}_r(t) =& C\mathbf{x}_r(t),\end{aligned} \tag{2}$$

where $\mathrm{sat}(\mathbf{u}_r(t), \mathbf{u}^*) = \mathbf{u}_r(t)$ *for any* $t \in [0,\ T_f]$. □

Remark 3 *Assumption 2 is sometimes called the model matching condition. It should be noted that in the convergence analysis of ILC, two types of convergences are widely used: one is for output tracking* $(\mathbf{y}_i(t) \to \mathbf{y}_r(t))$ *and the other is for input tracking* $(\mathbf{u}_i(t) \to \mathbf{u}_r(t))$. *Usually, the output tracking requires less knowledge of the system or a relatively weaker set of assumptions. In contrast, the input tracking requires Assumption 2 and a stronger resetting condition as shown in Assumption 3. It should be noted that when the input saturation is considered, Assumption 2 is always needed to define reachable trajectory.* ◦

Assumption 3 *We assume that the system performs the task repetitively over a finite interval of time* $[0,\ T_f], 0 < T_f < \infty$, *and satisfies the identical initial condition (i.i.c), expressed as* $\mathbf{x}_i(0) = \mathbf{x}_r(0),\ \forall i \in \mathcal{N}$. □

As discussed in Remark 3, this resetting condition is needed when input tracking is considered (see for example, in [28]).

Assumption 4 *It is assumed that system* (1) *has a relative degree* [2] *of one. For simplicity of presentation, it is assumed that* $CB > 0$. □

It is possible to relax Assumption 4 as the same analysis can be extended to systems with a higher relative degrees.

The tracking error at the i^{th} iteration is defined as:

$$\mathbf{e}_i(t) \triangleq \mathbf{y}_r(t) - \mathbf{y}_i(t). \tag{3}$$

The control objective is to find the control input $\mathbf{u}_i(t)$ for the system (1) with a saturation constraint such that when the task is repeated, the output error $\mathbf{e}_i(t)$ uniformly converges to 0. More precisely, $\forall\epsilon > 0$, $\exists\ N^* = N(\epsilon)$ such that $\forall i \ge N^*$, the tracking error satisfies : $|\mathbf{e}_i(t)| \le \epsilon,\ \forall t \in [0,\ T_f]$.

2.2 Controller Design

In this section, a standard design of feedback-based ILC scheme is reviewed where input saturation is not taken into account. Then a novel ILC algorithm is proposed to

[1] A MIMO square system is defined as a system which has the same dimensions for the input and output vectors

[2] "If the relative degree of system (1) is r, then $CB = CAB = \cdots = CA^{r-2}B = \mathbf{0}$ and $CA^{r-1}B \ne \mathbf{0}$" [19, p.387].

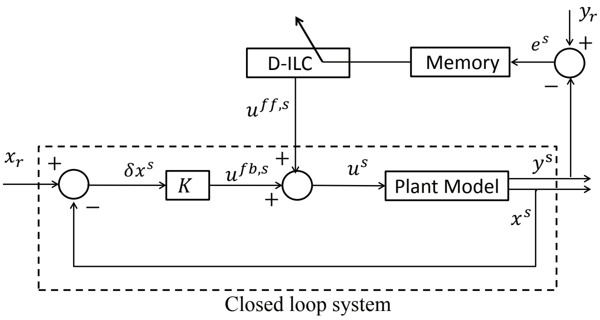

Fig. 1. Block diagram of D-type ILC with a stabilising feedback controller.

handle the input saturation. The schematics of the standard feedback-based ILC is shown in Fig.1.
The control law for the i^{th} iteration is generated from a dual controller design which is composed of a stabilising feedback controller and a feed-forward ILC. The state feedback is designed to ensure a good tracking performance of output and the state whereas the ILC is expected to iteratively find the desired control input for the reference trajectory. Without any loss of generality, we assume a stabilising feedback controller of the following form,

$$\mathbf{u}_i^{fb,s}(t) = K\left(\mathbf{x}_r(t) - \mathbf{x}_i^s(t)\right), \ \forall t \in [0, \ T_f], \tag{4}$$

where $K \in \mathcal{R}^{m\times n}$ is the feedback gain such that the matrix $(A - BK)$ is Hurwitz. If (A, B) are known, many techniques such as LQR, H_∞ and pole-placement can be used to design this feedback gain based on different control objectives in the time-domain. If (A, B) are unknown, some high-gain K can ensure the stability of the closed loop. When saturation happens, such a high gain controller might not work well. Here the superscript s is used to uniquely represent the signals from the feedback-based ILC scheme without considering the input saturation.

Remark 4 *The use of state feedback makes it easier to extend the dual control loop (or feedback-based ILC) to nonlinear time-invariant or time-varying systems. The role of introducing state feedback is two-fold. Firstly, in the LTI case, the role of state feedback is to ensure the good transient response in iteration-domain. For nonlinear cases, the state feedback can be used to ensure some given boundedness of the state trajectories. Once the state is bounded, it is possible to design ILC algorithms for nonlinear systems that are locally Lipschitz continuous (or global Lipschitz continuous when the state is already bounded) [26]. Secondly, the state feedback provides robustness with respect to modeling uncertainties and a large class of non-repetitive disturbances.* ◦

For the feedback-based ILC system shown in Fig.1, the total input is given by

$$\mathbf{u}_i^s(t) = \mathbf{u}_i^{ff,s}(t) + \mathbf{u}_i^{fb,s}(t), \tag{5}$$

where $\mathbf{u}_i^{ff,s}(t)$ represents the feed-forward input from D-type ILC given by (7).

Remark 5 *Usually, the feed-forward ILC is designed to track the reference output trajectory $y_r(t)$. In this paper both tracking error in terms of output (feed-forward) and the state (feedback) are used. In many applications such as robotic systems, the state can be computed directly from the output. For example, when the position is measured, the velocity can be computed from the position signals. If the task is repeatable, the desired input can be learned by using ILC.* ◦

The error dynamics of the closed loop system can be written in the form:

$$\begin{aligned}\delta\dot{\mathbf{x}}_i^s(t) =& (A - BK)\,\delta\mathbf{x}_i^s(t) + B\delta\mathbf{u}_i^{ff,s}(t)\\ \delta\mathbf{y}_i^s(t) =& C\delta\mathbf{x}_i^s(t),\end{aligned} \tag{6}$$

where $\delta\mathbf{x}_i{}^s = \mathbf{x}_r - \mathbf{x}_i^s$ and $\delta\mathbf{u}_i^{ff,s} = \mathbf{u}_r - \mathbf{u}_i^{ff,s}$.
The D-type ILC is given by

$$\begin{aligned}\mathbf{u}_{i+1}^{ff,s}(t) &= \mathbf{u}_i^{ff,s}(t) + q\Gamma\dot{\mathbf{e}}_i^s(t),\\ \mathbf{u}_1^{ff,s}(t) &= 0, \ t \in [0, \ T_f], \ i = 1, 2, \cdots,\end{aligned} \tag{7}$$

where $\mathbf{e}_i^s(t) \triangleq \mathbf{y}_r(t) - \mathbf{y}_i^s(t)$, $\Gamma \in \mathcal{R}^{m\times m}$ is a positive definite matrix gain, $q \in \mathcal{R}$ is the learning rate.

Remark 6 *One of the underlying assumptions in ILC design is that the actual model information (A, B, C) is unknown to the designer and only the nominal model of the plant (A_0, B_0, C_0) is available. The ILC update law* (7) *has two parameters for tuning– the learning rate q and matrix gain Γ– as compared with the conventional updating law [28]. This allows the designer to calculate Γ as the inverse of the nominal value of matrix C_0B_0 and tune the parameter q such that the convergence condition* (8) *is satisfied when implemented with the real system.* ◦

The sufficient condition for the convergence of the closed loop system (6) is

$$|I_m - q\Gamma CB| < 1, \tag{8}$$

which does not depend on closed loop matrix $(A - BK)$. Under Assumption 4 and a weaker Assumption 3, by using CM method,it can be shown that the λ -norm of the tracking error converges [28]. By using the norm equivalence condition, the supremum norm also convergences. This convergence condition (8) gives the flexibility to design the tuning parameter Γ and learning rate q without the knowledge of the system matrix A. It is noted that the ILC algorithm (7) has an integral format along the iteration-domain. Thus it is quite likely when the iteration number goes to infinity, the ILC control input might be unbounded at some time instants. Moreover, the existence of feedback (4) can also cause undesirable performance in time-domain when input saturation is considered. The question is - How to modify the feedback-based ILC consisting of ((4),(5), and (7)) such that the

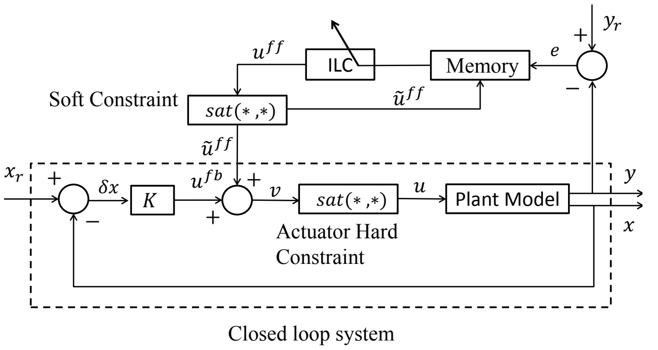

Fig. 2. Block diagram of a proposed dual controller system with input saturation.

tracking error converges in the presence of input saturation under the Assumptions 1 – 4?. Similar to standard anti-windup idea in classic PI control design [11], the main idea of this work is to design feedback-based ILC without considering input saturation while an novel structure is proposed to handle input saturation.

3 Proposed Control Law

The proposed control structure that incorporates the actuator saturation is shown in Fig. 2. It has two saturation constraints – the first saturation function is the soft constraint that stems from the ILC controller where as the second saturation function is the hard limitation from the actuator input to plant. It is assumed that both saturation functions have the same limit, represented by $\mathbf{u}^*$.
The control input is given by

$$\begin{aligned}\mathbf{u}_i(t) =& sat(\mathbf{v}_i(t),\ \mathbf{u}^*)\\ \mathbf{v}_i(t) =& \widetilde{\mathbf{u}}_i^{ff}(t) + \mathbf{u}_i^{fb}(t),\ \forall t \in [0,\ T_f],\end{aligned} \tag{9}$$

where $\widetilde{\mathbf{u}}_i^{ff}(t) = \text{sat}(\mathbf{u}_i^{ff}(t),\ \mathbf{u}^*)$ represents the modified input from the ILC control law given by (10) and $\mathbf{u}_i^{fb}(t)$ is the stabilising feedback control given by (11).
The proposed ILC algorithm is given by:

$$\mathbf{u}_{i+1}^{ff}(t) = \text{sat}(\mathbf{u}_i^{ff}(t),\ \mathbf{u}^*) + q\Gamma\dot{\mathbf{e}}_i(t),\ \mathbf{u}_1^{ff}(t) = 0, \tag{10}$$

$$\mathbf{u}_i^{fb}(t) = K\left(\mathbf{x}_r(t) - \mathbf{x}_i(t)\right),\ \forall t \in [0,\ T_f],\ i \in \mathcal{N}. \tag{11}$$

The ILC update law (10) is a modified form of a standard D-type ILC control law where an input saturation constraint is included to account for the hard limitations from the actuator. The saturation function in the update law plays a critical role in the iteration-domain performance. A saturation function could deteriorate the system performance and even lead to unstable performance. This paper analyse the convergence of the proposed ILC for a system with hard constraints from the actuator in the presence of a closed loop feedback controller.

Remark 7 *Generally, ILC does not require the full state information for the perfect output tracking. The ILC update law,* (10) *is based on the output tracking error whereas a stabilising state feedback is used. This dual controller design is widely used in the literature [2, 21, 22] without considering the hardware constraints from actuators.*
ILC (7) *can track the desired output trajectory when Assumptions 1–4 are satisfied. However, when the feedback control law* (4) *is used, it may also lead to saturation in the time-domain. Therefore, incorporating feedback into ILC design will make the convergence analysis difficult due to the existence of input saturation.* ◦

Proposition 1 is used to show that at the first iteration iteration, as $\mathbf{u}^{ff} = 0$, the linear state feedback law (11) can ensure the boundedness of trajectories by using feedback control law in the presence of input saturation. Even though the saturation can happen, the closed loop trajectories are still bounded over a finite time interval $[0, T_f]$. Again, it is possible that the feed-forward ILC could saturate the system, however because of Assumption 3 and finite time nature of the task, the trajectories are always bounded. This is discussed in Remark 9.

Proposition 1 *For a given time interval* $[0, T_f]$*, for the system* (1) *with control laws* (9) *–* (11) *satisfying the Assumptions* (1)*–* (4)*, all the trajectories of the system will stay in a compact set* $\mathcal{D}$ *when the feed-forward control input* $\mathbf{u}^{ff} = 0$.

Proof: When the feed-forward control input is zero, the system dynamics can be written in the form,

$$\delta\dot{\mathbf{x}} = (A - BK)\delta\mathbf{x} + B\mathbf{u}_r - B\left(K\delta\mathbf{x} - \text{sat}(K\delta\mathbf{x},\ \mathbf{u}^*)\right). \tag{12}$$

Consider a Lyapunov candidate $V = \frac{1}{2}\delta\mathbf{x}^{\intercal}\delta\mathbf{x}$. The derivative of V along the trajectories of (12) results in

$$\begin{aligned}\dot{V} =& \delta\mathbf{x}^{\intercal}(A - BK)\delta\mathbf{x} + \delta\mathbf{x}^{\intercal}B\mathbf{u}_r\\ &+ \delta\mathbf{x}^{\intercal}B\left(K\delta\mathbf{x} - \text{sat}(K\delta\mathbf{x},\ \mathbf{u}^*)\right).\end{aligned} \tag{13}$$

Obviously, K is chosen to have $A - BK$ as a Hurwitz matrix. There exists a positive constant L_{c_1} such that

$$|K\delta\mathbf{x} - \text{sat}(K\delta\mathbf{x},\ \mathbf{u}^*)| \le L_{c_1}.\,|\delta\mathbf{x}| \tag{14}$$

This leads to

$$\begin{aligned}\dot{V} &\le -\eta\,|\delta\mathbf{x}|^2 + |B|\,|\delta\mathbf{x}|\,|\mathbf{u}_r| + |B|\,L_{c_1}\,|\delta\mathbf{x}|^2\\ &\le -2\left(\eta - |B|\,L_{c_1}\right)V + |B|\,|\mathbf{u}_r|\sqrt{2V},\end{aligned} \tag{15}$$

for some $\eta > 0$. Due to Assumption 3, $V(0) = 0$, in the worst case, this Lyapunov function grows exponentially [14, Lemma 2.5 (Comparison Lemma)]. Therefore, it is possible to conclude that the trajectories are bounded for any $t \in [0,\ T_f]$. □

Remark 8 *Proposition 1 shows that when there is no feed-forward ILC, with input saturation, the feedback control law can ensure the boundedness of trajectories in a* *finite time interval* $[0,\ T_f]$*. Of course, as the system is controllable, by selecting the compact set of initial condition, it is possible to find a suitable feedback gain* K *such that the input saturation will not happen for this set of initial condition. Usually such feedback gain matrix cannot be very large. However, as the precise model of the system is unknown, it is hard to find such a small matrix* K*. In many applications, when the precise model is not completely known, a controller with a sufficiently high gain always work. But it will lead to input saturation. However, even if the input saturates, the closed loop trajectories will be still bounded over a finite time interval.* ◦

Remark 9 *When the feed-forward ILC exists, the closed loop dynamics can be written in the form,*

$$\delta\dot{\mathbf{x}} = (A - BK)\delta\mathbf{x} + B\delta\tilde{\mathbf{u}}^{ff} + B\left(\mathbf{v} - \mathrm{sat}(\mathbf{v},\ \mathbf{u}^*)\right) \tag{16}$$

Similar to the proof of Proposition 1, using the same Lyapunov candidate, it leads to

$$\dot{V} \le -\eta\left|\delta\mathbf{x}\right|^2 + |B|\left|\delta\mathbf{x}\right|\left|\delta\tilde{\mathbf{u}}^{ff}\right| + |B|\, L_{c_1}\left|\delta\mathbf{x}\right|^2, \tag{17}$$

for some $\eta > 0$*. As long as* $\left|\delta\tilde{\mathbf{u}}^{ff}\right| \in \mathcal{L}^2$*, it is not difficult to show that the trajectories of closed loop (feedback and feed-forward ILC) is uniformly bounded over a finite time interval* $[0,\ T_f]$. ◦

4 Convergence analysis

The main result is stated in Theorem 1.

Theorem 1 *Under the Assumptions 1–4, the system* (1) *with the control laws* (9),(10),(11) *satisfying the convergence condition* (8) *can*

(1) achieve zero output tracking error such that $\mathbf{e}_i(t)$ *converges to zero uniformly as* $i \to \infty$ *;*
(2) achieve zero state tracking error in the sense that $\delta\mathbf{x}_i(t)$ *converges to zero uniformly as* $i \to \infty$*;*
(3) ensure uniform boundedness of $\mathbf{u}_i^{ff}(t)$ *for all* $i \in \mathcal{N}$*;*
(4) and the feed forward input $\mathbf{u}_i^{ff}(t)$ *converges to reference input* $\mathbf{u}_r(t)$ *in* $\mathcal{L}^2$ *norm as* $i \to \infty$*.*

Proof: The proof consists of three main sections. Section 4.1 establishes the boundedness of the state and the output signals. Section 4.2 derives the difference of energy function and finally, Section 4.3 proves the convergence.
For the proof of convergence, we propose a new time weighted CEF, E_i. For any given $\lambda > 0$, E_i is given by:

$$\begin{aligned} E_i(t) =& \frac{1}{2}e^{-\lambda t}\delta\mathbf{x}_{i-1}^{\intercal}(t)\delta\mathbf{x}_{i-1}(t) \\ &+ \int_0^t e^{-\lambda\tau}\delta\mathbf{u}_i^{ff\intercal}(\tau)\delta\mathbf{u}_i^{ff}(\tau)d\tau, \\ &\forall t \in [0, T_f], i \in \mathcal{N}, \mathbf{x}_0(t) = 0, \end{aligned} \tag{18}$$

where $\delta\mathbf{x}(t) = \mathbf{x}_r(t) - \mathbf{x}(t)$ and $\delta\mathbf{u}^{ff}(t) = \mathbf{u}_r(t) - \mathbf{u}^{ff}(t)$. This CEF incorporates a time weighted state tracking error and $\mathcal{L}_e^2$ norm of the control input error. The first term is similar to a Lyapunov function which deals with the state tracking error of the previous iteration. The second term is equivalent to the $\mathcal{L}^2$ norm of the control input from the ILC. The intention of this proof is to show that $\Delta E_{i+1}(t)$ (19) is non-positive and $E_1(t)$ is bounded which leads to the convergence of output tracking and state tracking error as well as the convergence of feed-forward control input in $\mathcal{L}^2$ norm sense.
For convenience, we omit the notation t from the control signals in the following sections whenever appropriate.

4.1 Boundedness property

It is noted that the CEF at the current iteration E_i has the $\delta\mathbf{x}_{i-1}$ and $\mathcal{L}^2$ norm of $\delta\mathbf{u}_i$. At the first iteration, Proposition 1 show that the $\delta\mathbf{x}_1$ is bounded. Note that E_1 is bounded, the boundedness of $\delta\mathbf{x}_1$ is used to shows that $E_2 - E_1$ is non-increasing for any time instant in $[0, T_f]$. While E_2 is bounded, it indicates that $\delta\mathbf{u}_2$ is in $\mathcal{L}^2$. Remark 9 indicates that $\delta\mathbf{x}_2$ will be bounded. If it can be show that $E_{i+1} \le E_i$, then $\delta\mathbf{u}_i$ will have a uniform bound in $\mathcal{L}^2$, leading to a uniform boundedness of $\delta\mathbf{x}_i$.

Next, it will show that if E_i is bounded, then $E_{i+1} \le E_i$.

4.2 Difference of composite energy function

In this section of the proof we show that the CEF is non-increasing in the iteration-domain. The difference of energy function between two iterations is :

$$\begin{aligned} \Delta E_{i+1} =& E_{i+1} - E_i \\ =& \frac{1}{2}e^{-\lambda t}\left(\left|\delta\mathbf{x}_i\right|^2 - \left|\delta\mathbf{x}_{i-1}\right|^2\right) \\ &+ \int_0^t e^{-\lambda\tau}\left(\left|\delta\mathbf{u}_{i+1}^{ff}\right|^2 - \left|\delta\mathbf{u}_i^{ff}\right|^2\right)d\tau. \end{aligned} \tag{19}$$

In the following part of this proof, we establish a few relations that are beneficial in concluding the proof of the theorem.
Firstly, from the ILC update law (10) we have

$$\delta\mathbf{u}_{i+1}^{ff} = \delta\tilde{\mathbf{u}}_i^{ff} - q\Gamma\dot{\mathbf{e}}_i, \tag{20}$$

where $\delta\tilde{\mathbf{u}}_i^{ff} = \mathbf{u}_r - \mathrm{sat}(\mathbf{u}_i^{ff}, \mathbf{u}^*)$. Using the plant dynamics (1) and reference dynamics (2), we get

$$q\Gamma\dot{\mathbf{e}}_i = q\Gamma CA\delta\mathbf{x}_i + q\Gamma CB\delta\mathbf{u}_i. \tag{21}$$

Substituting (21) back into (20) yields,

$$\delta\mathbf{u}_{i+1}^{ff} = \mathrm{P}\delta\tilde{\mathbf{u}}_i^{ff} + \mathbf{z}_i, \tag{22}$$

where $\mathrm{P} = (I_m - q\Gamma CB)$ and $\mathbf{z}_i = -q\Gamma CA\delta\mathbf{x}_i + q\Gamma CB(\delta\tilde{\mathbf{u}}_i^{ff} - \delta\mathbf{u}_i)$. Using (22), we can show that

$$
\begin{aligned}
&\delta\mathbf{u}_{i+1}^{ff\intercal}\delta\mathbf{u}_{i+1}^{ff} - \delta\widetilde{\mathbf{u}}_i^{ff\intercal}\delta\widetilde{\mathbf{u}}_i^{ff} \\
&\quad = -\delta\widetilde{\mathbf{u}}_i^{ff\intercal}(I_m - \mathrm{P}^\intercal\mathrm{P})\delta\widetilde{\mathbf{u}}_i^{ff} + |\mathbf{z}_i|^2 + 2\mathbf{z}_i^\intercal\mathrm{P}\delta\widetilde{\mathbf{u}}_i^{ff} \\
&\quad \le -\lambda_p\left|\delta\widetilde{\mathbf{u}}_i^{ff}\right|^2 + |\mathbf{z}_i|^2 + 2|\mathbf{z}_i|\left|\mathrm{P}\delta\widetilde{\mathbf{u}}_i^{ff}\right|, \qquad (23)
\end{aligned}
$$

where $\lambda_p = \lambda_{max}(I_m - \mathrm{P}^\intercal\mathrm{P})$.
It is possible to find $\Gamma > 0$ and $q < 1$ such that $|P| < 1$ (as in Equation (8)) which leads to $I_m - \mathrm{P}^\intercal\mathrm{P} > 0$ and therefore $\lambda_p > 0$.
Secondly, using Lemma 2 on $\left|\delta\widetilde{\mathbf{u}}_i^{ff} - \delta\mathbf{u}_i\right|$ yields,

$$
\begin{aligned}
\left|\delta\widetilde{\mathbf{u}}_i^{ff} - \delta\mathbf{u}_i\right| &= \left|\mathbf{u}_i - \widetilde{\mathbf{u}}_i^{ff}\right| = \left|\mathrm{sat}(\mathbf{v}_i,\mathbf{u}^*) - \mathbf{v}_i + \mathbf{u}_i^{fb}\right| \\
&\le |\mathrm{sat}(\mathbf{v}_i,\mathbf{u}^*) - \mathbf{v}_i| + \left|\mathbf{u}_i^{fb}\right| \\
&\le 2\left|\mathbf{u}_i^{fb}\right| = 2|K||\delta\mathbf{x}_i|. \qquad (24)
\end{aligned}
$$

With the aid of (24), the second term in (23) yields

$$
|\mathbf{z}_i| = \left|-q\Gamma CA\delta\mathbf{x}_i + q\Gamma CB(\delta\widetilde{\mathbf{u}}_i^{ff} - \delta\mathbf{u}_i)\right| \le \lambda_f|\delta\mathbf{x}_i|, \qquad (25)
$$

where $\lambda_f = |q\Gamma CA| + 2|q\Gamma CB||K|$.
Thirdly, by using completion of squares, we can show that there exists an $\alpha > 0$ such that:

$$
\begin{aligned}
&|\delta\mathbf{x}_i|\left|\delta\widetilde{\mathbf{u}}_i^{ff}\right| \\
&= \sqrt{\alpha}|\delta\mathbf{x}_i|\frac{1}{\sqrt{\alpha}}\left|\delta\widetilde{\mathbf{u}}_i^{ff}\right| \le \frac{\alpha}{2}|\delta\mathbf{x}_i|^2 + \frac{1}{2\alpha}\left|\delta\widetilde{\mathbf{u}}_i^{ff}\right|^2. \qquad (26)
\end{aligned}
$$

Lastly, using Lemma 2 on $|\delta\mathbf{u}_i|$ yields:

$$
\begin{aligned}
|\delta\mathbf{u}_i| &= |\mathbf{u}_r - \mathrm{sat}(\mathbf{v}_i,\mathbf{u}^*)| \\
&= |\mathbf{u}_r - \mathbf{v}_i - (sat(\mathbf{v}_i,\mathbf{u}^*) - \mathbf{v}_i)| \\
&\le \left|\mathbf{u}_r - \widetilde{\mathbf{u}}_i^{ff} - \mathbf{u}_i^{fb}\right| + |\mathrm{sat}(\mathbf{v}_i,\mathbf{u}^*) - \mathbf{v}_i| \\
&\le \left|\delta\widetilde{\mathbf{u}}_i^{ff}\right| + 2|K||\delta\mathbf{x}_i|. \qquad (27)
\end{aligned}
$$

Using Lemma 1 on $\left|\delta\mathbf{u}_{i+1}^{ff}\right|^2 - \left|\delta\mathbf{u}_i^{ff}\right|^2$ followed by substituting (25) yields

$$
\begin{aligned}
\left|\delta\mathbf{u}_{i+1}^{ff}\right|^2 - \left|\delta\mathbf{u}_i^{ff}\right|^2 &\le \left|\delta\mathbf{u}_{i+1}^{ff}\right|^2 - \left|\delta\widetilde{\mathbf{u}}_i^{ff}\right|^2 \\
&\le -\left(\lambda_p - \frac{1}{\alpha}\lambda_f|\mathrm{P}|\right)\left|\delta\widetilde{\mathbf{u}}_i^{ff}\right|^2 \\
&\quad + \left(\lambda_f^2 + \alpha\lambda_f|\mathrm{P}|\right)|\delta\mathbf{x}_i|^2. \qquad (28)
\end{aligned}
$$

Next, we make use of (23)- (28) to find the difference of CEF. Using the Assumption 3, the first part of first term in equation (19) can be expanded, followed by substituting (27) and (26), it yields

$$
\begin{aligned}
\frac{1}{2}e^{-\lambda t}|\delta\mathbf{x}_i|^2 &= -\frac{\lambda}{2}\int_0^t e^{-\lambda\tau}|\delta\mathbf{x}_i|^2 d\tau + \int_0^t e^{-\lambda\tau}\delta\mathbf{x}_i^\intercal\delta\dot{\mathbf{x}}_i d\tau \\
&= -\frac{\lambda}{2}\int_0^t e^{-\lambda\tau}|\delta\mathbf{x}_i|^2 d\tau \\
&\quad + \int_0^t e^{-\lambda\tau}\delta\mathbf{x}_i^\intercal(A\delta\mathbf{x}_i + B\delta\mathbf{u}_i)d\tau \\
&\le -\left(\frac{\lambda}{2} - \lambda_k\right)\int_0^t e^{-\lambda\tau}|\delta\mathbf{x}_i|^2 d\tau \\
&\quad + \frac{1}{2\alpha}|B|\int_0^t e^{-\lambda\tau}\left|\delta\widetilde{\mathbf{u}}_i^{ff}\right|^2 d\tau. \qquad (29)
\end{aligned}
$$

where $\lambda_k = |A| + 2|B||K| + \frac{\alpha}{2}|B|$.
The difference of energy equation can be expanded by substituting (29) and (28) into (19) as well as considering the positiveness of $|\delta\mathbf{x}_{i-1}|^2$ in (19). This results in the inequality relation:

$$
\begin{aligned}
\Delta E_{i+1} \le &- L_\lambda\int_0^t e^{-\lambda\tau}|\delta\mathbf{x}_i|^2 d\tau \\
&- L_\alpha\int_0^t e^{-\lambda\tau}\left|\delta\widetilde{\mathbf{u}}_i^{ff}\right|^2 d\tau, \qquad (30)
\end{aligned}
$$

where $L_\lambda = \frac{\lambda}{2} - \lambda_k - \lambda_f^2 - \alpha\lambda_f|P|$; $L_\alpha = \lambda_p - \frac{1}{2\alpha}(2\lambda_f|\mathrm{P}| + |B|)$. There exists $\alpha > 0$ and $\lambda > 2\left(\lambda_k + \lambda_f^2 + \alpha\lambda_f|\mathrm{P}|\right)$ such that $L_\alpha > 0$ and $L_\lambda > 0$, which leads to $\Delta E_{i+1}(t) \le 0$. Therefore we can conclude that E_{i+1} is non-increasing along the iteration axis.

4.3 Convergence property

In this section, point-wise convergence is attained followed by the uniform convergence property. For point wise convergence, we show that $\delta\mathbf{x}_i(t) \to \mathbf{0}$ as $i \to \infty$ and $\delta\mathbf{u}_i^{ff}(t) \to \mathbf{0}$ as $i \to \infty$ point-wisely.

Firstly, $E_{i+1}(t) = E_1(t) + \sum_{j=1}^{i}\Delta E_{j+1}(t)$, $\forall t \in [0,\ T_f]$. As $\Delta E_{i+1}(t) \le 0$, $E_{i+1}(t)$ is non increasing along the iteration axis. This means that $\lim_{i\to\infty}E_{j+1}(t)$ exists and $\lim_{i\to\infty}E_{i+1}(t) = E_1(t) + \lim_{i\to\infty}\sum_{j=1}^{i}\Delta E_{i+1}(t) \le E_1(t)$. This series will bounded if $E_1(t)$ is finite. As $\mathbf{x}_0 = \mathbf{0}$ and $\mathbf{u}_1^{ff} = \mathbf{0}$, $E_1(t) = \frac{1}{2}e^{-\lambda t}|\mathbf{x}_r|^2 + \int_0^t e^{-\lambda\tau}|\mathbf{u}_r|^2 d\tau$. As per Assumption 2, $\mathbf{x}_r$ and $\mathbf{u}_r$ are uniformly continuous functions, hence $E_1(t)$ is finite for $t \in [0,\ T_f]$. In addition, $\sum_{j=1}^{i}\Delta E_{i+1}(t)$ converges. From the convergence theorem [23], as the sum of the series converges to zero, the series converges, leading to $\lim_{i\to\infty}\Delta E_{i+1}(t) = 0, \forall t \in [0,\ T_f]$. Specifically, when $t = T_f$, (30) yields:

$$
\begin{aligned}
&\lim_{i\to\infty}\int_0^{T_f} e^{-\lambda\tau}\left[L_\lambda|\delta\mathbf{x}_i|^2 + L_\alpha\left|\delta\widetilde{\mathbf{u}}_i^{ff}\right|^2\right]d\tau = 0, \\
&\lim_{i\to\infty}\|\delta\mathbf{x}_i\|_{\mathcal{L}_e^2} = 0 \text{ and } \lim_{i\to\infty}\left\|\delta\widetilde{\mathbf{u}}_i^{ff}\right\|_{\mathcal{L}_e^2} = 0. \qquad (31)
\end{aligned}
$$

Moreover, the uniform boundedness of $\delta\dot{\mathbf{x}}_i(t)$ ensures the uniform continuity of $\delta\mathbf{x}_i(t)$. This leads to the following conclusions

(1) the state tracking error converges to 0 uniformly,
(2) consequently the output tracking error converges uniformly,
(3) the feed-forward control input converges uniformly in $\mathcal{L}^2$ norm sense as $i \to \infty$.

This completes the proof. □

Remark 10 *Although Theorem 1 states that* $\lim_{i\to\infty} \mathbf{u}_i^{ff}(t) = \mathbf{u}_r(t)$*, we can conclude that* $\lim_{i\to\infty} \mathbf{u}_i(t) = \mathbf{u}_r(t)$ *as the feedback disappears as the iteration number approaches to infinity.* ∘

Remark 11 *Note that* $\lambda_k, \lambda_f, \mathrm{P}, B$ *are the system parameters. The value of* λ *in the CEF plays an important role in the proof. By selecting a suitable* λ*, it is possible to ensure that input-output mapping will dominate the convergence analysis. If the condition* (8) *is satisfied, this CEF can ensure the convergence of tracking errors in terms of the input, the state, and the output.* ∘

Remark 12 *The CEF* (18) *has a unique form as it consists of the state tracking performance of last iteration and the current error in terms of feed-forward input signal with respect to the reference input. The CEF is a kind of energy function along both the time-domain and iteration-domain. The convergence is related to the performance when the iteration number goes to infinity. Thus the CEF can have the flexibility to deal with signals at different iterations. If the CEF converges, we can obtain the convergence of tracking error (both in state and output).* ∘

5 An Illustrative Example

In order to demonstrate the effectiveness of the proposed controller, we consider (1) with system matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1.5 \\ 0 & 0 & 0 & 1 \\ 0 & -0.75 & 0 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 2.5 & 0.45 \\ 0 & 0 \\ 0.45 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (32)$$

The system model (32) is a linearised model of a two link robot manipulator at the joint space. The reference output $\mathbf{y}_r(t) = [r(t), r(t)]^\intercal$ is the reference velocity of joint angles where

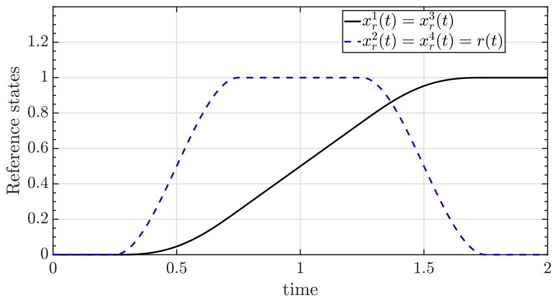

Fig. 3. State reference trajectories

$$r(t) = \begin{cases} 0, & 0 \le t \le \frac{T_f}{8} \\ \phi_1(t), & \frac{T_f}{8} < t \le \frac{3T_f}{8} \\ 1, & \frac{3T_f}{8} < t \le \frac{5T_f}{8} \\ \phi_2(t), & \frac{5T_f}{8} < t \le \frac{7T_f}{8} \\ 0, & \frac{7T_f}{8} < t \le T_f \end{cases}, \qquad (33)$$

$\phi_1(t) = \frac{3}{\tau^2}\left(t - \frac{T_f}{8}\right)^2 - \frac{2}{\tau^3}\left(t - \frac{T_f}{8}\right)^3$, $\phi_2(t) = 1 - \frac{3}{\tau^2}\left(t - \frac{5T_f}{8}\right)^2 + \frac{2}{\tau^3}\left(t - \frac{5T_f}{8}\right)^3$ and $\tau = \frac{T_f}{4}$. For $T_f = 2$, the reference state x_r^1, x_r^2, x_r^3 and x_r^4 are plotted in Fig. 3 which satisfies the Assumption 2.

The matrix A in (32) is not Hurwitz, but (A, B) is controllable. The feed-forward is designed such that $\Gamma = (CB)^{-1}$ and $q = 0.95$, which satisfies the convergence condition (8) as well as $\lambda_p > 0$. The derivative of output tracking error is calculated by using the backward difference method followed by a noise filtering using a third order Butterworth filter of cut-off frequency $250Hz$. A time step of $0.001s$ is used in simulations.

The purpose of following simulations is to compare the transient performance in the iteration-domain, hence the supremum norm of tracking error and control input is plotted for different cases as explained in the following subsections.

5.1 Revisiting standard D-type ILC without hard input constraint

In this subsection, the responses of the system when implemented with a standard D-type ILC and feedback controller (as explained in Section 2.2) and D-type ILC with the feedback without any hard input saturation are discussed. Two simulations were performed with a standard D-ILC – one without a feedback and another with a feedback controller. The feedback gain is selected by placing the closed loop poles at $[-10, -5, -5 \pm 3j]$. The supremum norm of output tracking error is shown in Fig. 4. It can be seen that the use of feedback controller has improved the transient response in the iteration-domain (smaller overshoot in iteration domain). However, the

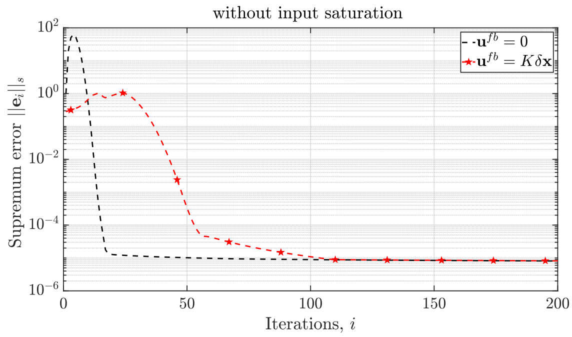

Fig. 4. Convergence of supremum norm of tracking error in iteration domain without hard input saturation

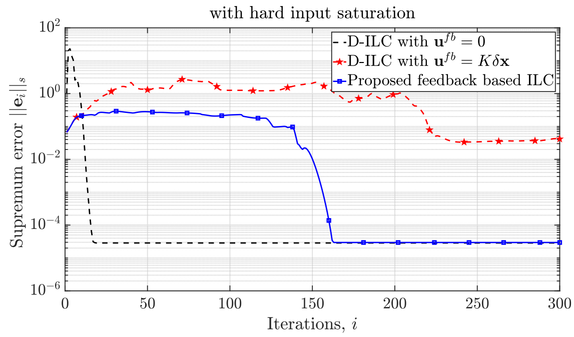

Fig. 5. Convergence of supremum norm of tracking error in iteration-domain (Simulation Set I)

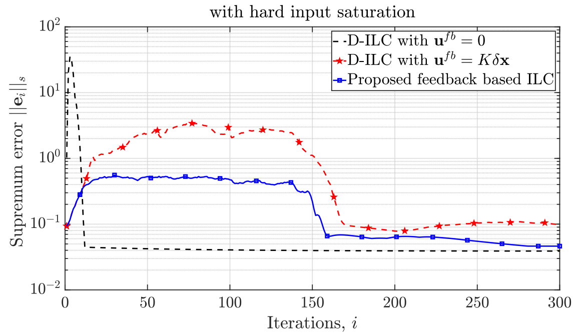

Fig. 6. Convergence of supremum norm of tracking error in iteration-domain (Simulation Set II)

convergence speed in terms of number of iterations has been affected in the case where feedback controller is used with the feed-forward ILC. This effect of feedback gain in the convergence of feed-forward ILC has been investigated in [20].

Next, the responses of the system (two cases) in presence of hard input saturation are investigated. Due to existence of input constraints, the feed-forward ILC with feedback cannot work in some cases.

5.2 ILC with hard input constraint

In this subsection, in order to test the robustness of the proposed method, two set of simulations were performed with the consideration of hard input constraint for feed-forward ILC, feed-forward ILC with feedback, and the proposed feedback-based ILC respectively. For the simulations, $\mathbf{u}^* = [2.1, \; 2.1]^\intercal$ is chosen as the saturation limit. The first set of simulations compare with standard D-type ILC with the proposed feedback-based ILC while the second set of simulations compare the standard D-type ILC with the proposed feedback-based ILC when there are white noises in measured state signals. For both the simulations, feedback gain K is chosen by placing the closed loop poles at $[-40, \; -30, \; -14 \pm 8j]$. This high gain feedback is chosen to handle the non-repeatable noises in the state measurements. The band limited white noises with a noise power of 10^{-6} has been added to all state trajectories in Simulation Set II.

Remark 13 *In this work, design of feedback is decoupled with feed-forward ILC in the presence of input saturation. How to achieve better performance in terms of maximum overshoot and convergence in iteration domain will be investigated in future when the knowledge of the system is available. In simulations, we only show that, without picking up an optimal gain, the proposed algorithm can still work better than other two algorithms.* ◦

The convergence of supremum norm of tracking error for Simulation Set-I and Simulation Set-II are shown in Fig. 5 and Fig. 6. Obviously, compared with the feed-forward ILC and feed-forward ILC with the feedback, the proposed feedback-based ILC can work very well in the presence of input saturation as well as the non-repeatable disturbances. It is noted that even when non-repeatable disturbances exist, the proposed method outperforms other two methods in terms of overshoot and convergence speed in iteration-domain for the chosen values of feedback gain. Different choice of feedback gains were also tested by placing the closed-loop eigenvalues. It is observed from a large number of simulations, if the input constraints are tight, the proposed control law usually works better. The simulation results demonstrate the effectiveness of the proposed feed-back ILC.

6 Conclusion

This paper presents a new feedback-based iterative learning control (ILC) algorithm that can handle input saturation. The design of the feedback and ILC is decoupled without considering input saturation, leading to a simple design procedure. A novel structure is constructed to handle the effect from input saturation. A new composite energy function is proposed to assist the convergence analysis of this new feedback-based ILC. Simulation results show the effectiveness of the proposed algorithm. Future work aims at extending the current control schematics to nonlinear systems with input and output constraints.

References

[1] B. Altin and K. Barton. Robust iterative learning for high precision motion control through $\mathcal{L}_1$ adaptive feedback. *Mechatronics*, 24(6):549–561, 2014.

[2] N. Amann, D. H. Owens, and E. Rogers. Iterative learning control using optimal feedback and feedforward actions. *International Journal of Control*, 65(2):277–293, 1996.

[3] N. Amann, D. H. Owens, E. Rogers, and A. Wahl. An H_∞ approach to linear iterative learning control design. *International Journal of Adaptive Control and Signal Processing*, 10(6):767–781, 1996.

[4] S. Arimoto, S. Kawamura, and F. Miyazaki. Bettering operation of robots by learning. *Journal of Robotic Systems*, 1(2):123–140, 1984.

[5] K. L. Barton and A. G. Alleyne. A cross-coupled iterative learning control design for precision motion control. *IEEE Transactions on Control Systems Technology*, 16(6):1218–1231, 2008.

[6] D. A. Bristow, M. Tharayil, and A. G. Alleyne. A survey of iterative learning control. *IEEE Control Systems*, 26(3):96–114, 2006.

[7] T.-Y. Doh. Robust iterative learning control with current feedback for uncertain linear systems. *International Journal of Systems Science*, 30(1):39–47, 1999.

[8] C. T. Freeman. Robust ILC design with application to stroke rehabilitation. *Automatica*, 81:270–278, 2017.

[9] J. E. Hauser. Learning control for a class of nonlinear systems. In *Decision and Control, 1987. 26th IEEE Conference on*, volume 26, pages 859–860. IEEE, 1987.

[10] Z. Hou, J.-X. Xu, and J. Yan. An iterative learning approach for density control of freeway traffic flow via ramp metering. *Transportation Research Part C: Emerging Technologies*, 16(1):71–97, 2008.

[11] T. Hu and Z. Lin. *Control Systems with Actuator Saturation: Analysis and Design*. Springer Science & Business Media, 2001.

[12] T.-J. Jang, C.-H. Choi, and H.-S. Ahn. Iterative learning control in feedback systems. *Automatica*, 31(2):243–248, 1995.

[13] S. Kawamura, F. Miyazaki, and S. Arimoto. Robot motion based on a learning method. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 18(1):126–134, 1988.

[14] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ 07458, 2 edition, 1996.

[15] T.-Y. Kuc, J. S. Lee, and K. Nam. An iterative learning control theory for a class of nonlinear dynamic systems. *Automatica*, 28(6):1215–1221, 1992.

[16] K. S. Lee, S. H. Bang, and K. S. Chang. Feedback-assisted iterative learning control based on an inverse process model. *Journal of Process Control*, 4(2):77–89, 1994.

[17] M. Mezghani, G. Roux, M. Cabassud, M. V. Le Lann, B. Dahhou, and G. Casamatta. Application of iterative learning control to an exothermic semibatch chemical reactor. *IEEE Transactions on Control Systems Technology*, 10(6):822–834, 2002.

[18] K. L. Moore. Iterative learning control: An expository overview. In *Applied and computational control, signals, and circuits*, pages 151–214. Springer, 1999.

[19] S. S. Sastry. *Nonlinear Systems: Analysis, Stability, and Control*, volume 10. Springer Science & Business Media, 2013.

[20] G. Sebastian, Y. Tan, D. Oetomo, and I. Mareels. Design of feedback gain in feedback-based iterative learning control. In *11th Asian Control Conference (ASCC 2017)*, pages 875–880, Dec 2017.

[21] J. Shi, F. Gao, and T.-J. Wu. Integrated design and structure analysis of robust iterative learning control system based on a two-dimensional model. *Industrial & Engineering Chemistry Research*, 44(21):8095–8105, 2005.

[22] J. Shi, F. Gao, and T.-J. Wu. Robust design of integrated feedback and iterative learning control of a batch process based on a 2D Roesser system. *Journal of Process Control*, 15(8):907–924, 2005.

[23] I. S. Sokolnikoff, R. M. Redheffer, and J. Avents. Mathematics of physics and modern engineering. *Journal of The Electrochemical Society*, 105(9):196C—-196C, 1958.

[24] M. X. Sun, D. W. Wang, and P. N. Chen. Repetitive learning control of nonlinear systems over finite intervals. *Science China Information Sciences*, 53(1):115–128, 2010.

[25] Y. Tan, H.-H. Dai, D. Huang, and J.-X. Xu. Unified iterative learning control schemes for nonlinear dynamic systems with nonlinear input uncertainties. *Automatica*, 48(12):3173–3182, 2012.

[26] Y. Tan, S. P. Yang, and J. X. Xu. On P-type iterative learning control for nonlinear systems without global Lipschitz continuity condition. In *American Control Conference (ACC), 2015*, pages 3552–3557. IEEE, 2015.

[27] O. Tutty, M. Blackwell, E. Rogers, and R. Sandberg. Iterative learning control for improved aerodynamic load performance of wind turbines with smart rotors. *IEEE Transactions on Control Systems Technology*, 22(3):967–979, 2014.

[28] J.-X. Xu and Y. Tan. *Linear and Nonlinear Iterative Learning Control*, volume 291. Springer, 2003.

[29] J.-X. Xu, Y. Tan, and T.-H. Lee. Iterative learning control design based on composite energy function with input saturation. *Automatica*, 40(8):1371–1377, aug 2004.

[30] J.-X. Xu, X.-W. Wang, and L. T. Heng. Analysis of continuous iterative learning control systems using current cycle feedback. In *American Control Conference, Proceedings of the 1995*, volume 6, pages 4221–4225. IEEE, 1995.

[31] W. Xu, B. Chu, and E. Rogers. Iterative learning control for robotic-assisted upper limb stroke rehabilitation in the presence of muscle fatigue. *Control Engineering Practice*, 31:63–72, 2014.

[32] S.-H. Zhou, J. Fong, V. Crocher, Y. Tan, D. Oetomo, and I. Mareels. Learning control in robot-assisted rehabilitation of motor skills–a review. *Journal of Control and Decision*, 3(1):19–43, 2016.