

On the Positive Output Controllability of Linear Time Invariant Systems

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Abstract

This paper considers the output controllability of autonomous linear control systems that are subject to non-negative input constraints. Based on the evaluation of the geometric properties of the system, necessary and sufficient conditions are proposed for the *positive output controllability* of continuous linear time invariant systems. To aid in the practical evaluation of positive output controllability, additional sufficient conditions are derived for which efficient numerical techniques exist. These conditions are evaluated over a set of numerical examples which support the theoretical results.

Key words: Linear systems; controllability; positive output controllability.

1 Introduction

The property of controllability, introduced by Kalman [1], evaluates the ability of a dynamic system to have its state driven from any initial state to any final state in a finite amount of time. The aim of studying the controllability properties of a dynamic system is to determine if a controller can be applied to generate a desired state space behaviour. For linear time invariant (LTI) systems, necessary and sufficient conditions have been identified [1, 2]. For nonlinear time invariant systems, linearisation has been used to obtain sufficient conditions for local controllability [3]. In addition, sufficient conditions have been proposed using Lie algebra for local controllability and/or the global controllability of some nonlinear systems [3, 4].

When constraints are imposed on either the system states or inputs, the effect of the constraints can alter the controllability conditions. This paper focuses on non-negative input constraints, which are motivated by engineered systems such as non-prehensile mechanisms [5], cable robots [6, 7], one way valves [8] and the antivibration control of pendulum systems [9]. This class of constraints have therefore been widely investigated resulting in different necessary and sufficient

conditions for controllability being identified for continuous [9–12] and discrete [13] LTI systems. Additionally, sufficient conditions for local positive controllability of nonlinear systems have been obtained [10, 14].

It is worthwhile to note that controllability is defined for states instead of outputs. In most engineering applications, tasks are defined for outputs, whose dimension can be much lower than that of the state. One example is the control of a multi-link cable driven manipulator, where the task is typically defined in terms of end effector pose, rather than the joint positions and velocities which can define the system's state [15]. Under such a situation, it is natural to consider output controllability (see for example, [16, 17] and references therein). In the evaluation of output controllability, necessary and sufficient conditions for LTI systems are well established [18]. For systems subject to non-negative input constraints there are no known results that consider output controllability.

In this paper, positive output controllability is defined for continuous LTI systems. Necessary and sufficient conditions for positive output controllability are derived. To more efficiently verify positive output controllability, some geometric sufficient conditions are proposed. These conditions are shown to be necessary and sufficient for two dimensional systems. The conditions are evaluated on numerical examples to support the theoretical results.

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2 Preliminaries

2.1 Notation

Denote the set of real numbers as \mathbb{R} , the complex numbers as \mathbb{C} , the square identity matrix with m rows as I_m and the zero matrix with m rows and n columns as $0_{(m \times n)}$. If the vector $\mathbf{x} = [x_1 \dots x_n]^T \in \mathbb{R}^n$ satisfies $x_i > 0$ ($x_i \geq 0$) for all $i \in \{1, \dots, n\}$, then \mathbf{x} is said to be positive (non-negative) and is denoted by $\mathbf{x} > \mathbf{0}$ ($\mathbf{x} \geq \mathbf{0}$). For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the inner product is denoted $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{x}$. For the vector $\mathbf{x} \in \mathbb{C}^n$, the complex conjugate is denoted $\bar{\mathbf{x}}$ and an orthogonal vector \mathbf{x}^\perp .

Let an unforced continuous LTI system be given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n, \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$. The unforced response of the system (1) is given by $\mathbf{x}(t) = e^{At}\mathbf{x}_0$. The eigenvalues of A are denoted by the set $\Lambda(A) = \Lambda^r(A) \cup \Lambda^c(A)$, where $\Lambda^r(A) \subseteq \mathbb{R}$ represents the i purely real eigenvalues and $\Lambda^c(A) \subseteq \mathbb{C}$ the remaining j eigenvalues such that $i + j = n$. Let the i real eigenvalues of $\Lambda^r(A)$ be defined such that the $k \leq i$ distinct real eigenvalues λ are arranged in the form $\lambda_1 > \dots > \lambda_k \in \Lambda^r$ and let the $l \leq \frac{j}{2}$ distinct real component ρ_j of the complex eigenvalues be arranged in the set $R^c(A) \subset \mathbb{R}$ such that $\rho_1 \geq \dots \geq \rho_l \in R^c$ for $j \in \{1, \dots, l\}$. The corresponding eigenvectors for the eigenvalue λ is given by $\varepsilon(\lambda)$ and the set of all eigenvectors for A is given by the eigenspace $\mathcal{E}(A)$.

2.2 Geometric Cone Theory

Definition 1 A set $\mathcal{X} \subseteq \mathbb{R}^n$ is said to be a cone if for all $\mathbf{x} \in \mathcal{X}$ and $\alpha \geq 0$, $\alpha\mathbf{x} \in \mathcal{X}$. The set is a convex cone if it is a cone and for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{x} + \mathbf{y} \in \mathcal{X}$ [19].

Definition 2 The extreme rays of the cone \mathcal{X} are the rays that cannot be expressed as a positive linear combination of other rays in \mathcal{X} [20].

Remark 1 Extreme rays form a positive linearly independent set that can provide a description of \mathcal{X} . An alternative description of \mathcal{X} can be provided using the matrix $G \in \mathbb{R}^{q \times n}$, where $\text{cone}(G) = \{\mathbf{x} \in \mathbb{R}^n \mid G\mathbf{x} \leq \mathbf{0}\}$. \circ

Definition 3 Let $K = [\mathbf{k}_1 \dots \mathbf{k}_m] \in \mathbb{R}^{n \times m}$ where n and m are positive integers. The image (or span) of the matrix K is defined as the set $\text{Im}(K) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \sum_{i=1}^m \alpha_i \mathbf{k}_i, \alpha_i \in \mathbb{R}\}$. The positive span of the matrix K is defined as the set $\text{span}_+(K) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \sum_{i=1}^m \alpha_i \mathbf{k}_i, \alpha_i \geq 0\}$.

Definition 4 Let $\mathcal{X} \subseteq \mathbb{R}^n$. The negative polar cone of the set \mathcal{X} , denoted \mathcal{X}^- , is the set of all $\mathbf{y} \in \mathbb{R}^n$ such that $\langle \mathbf{y}, \mathbf{x} \rangle \leq 0 \forall \mathbf{x} \in \mathcal{X}$ [19].

The negative polar cone and the positive span of a matrix always form convex cones by Definition 1.

2.3 Positive Invariance of Cones

Let $A \in \mathbb{R}^{n \times n}$ be a given state matrix of (1) and let λ represent an eigenvalue of A . The following definitions hold for λ and A .

Definition 5 A cone \mathcal{X} is positively invariant with respect to system (1) if $\forall t > 0$, $e^{At}\mathcal{X} \subseteq \mathcal{X}$ [21].

Definition 6 If a subspace $\mathcal{Y} \subseteq \mathbb{R}^n$ is positively invariant with respect to (1) then the subspace is said to be A -invariant and for all $\mathbf{x} \in \mathcal{Y}$, $A\mathbf{x} \in \mathcal{Y}$ [2].

Definition 7 The operating subspace $\mathcal{O}(\lambda)$ is the largest A -invariant subspace such that for all $\mathbf{x} \in \mathcal{O}(\lambda)$, there exists matrices $M(t), N(t) \in \mathbb{C}^{n \times n}$ such that $e^{At}\mathbf{x} = \left(e^{\lambda t} M(t) + e^{\bar{\lambda} t} N(t) \right) \mathbf{x}$.

Remark 2 The operational subspace $\mathcal{O}(\lambda)$ is equal to $\varepsilon(\lambda)$ if λ is real and $\text{rank}(\varepsilon(\lambda)) = m$, where m is the algebraic multiplicity of λ . If λ is complex, then $\mathcal{O}(\lambda)$ is the plane of oscillation and in the case of defective matrices it is given by the span of $\varepsilon(\lambda)$ and the generalised eigenvectors. \circ

Definition 8 Let $\mathcal{T} \subseteq \mathbb{R}^n$ be a positively invariant cone with respect to (1). The \mathcal{T} -dominant eigenvalue is the eigenvalue $\lambda^*(A, \mathcal{T})$ with largest real component such that \exists a positively invariant cone $\mathcal{T}_1 \subseteq (\mathcal{T} \cap \mathcal{O}(\lambda^*))$ with dimension greater than 0. The \mathcal{T} -dominant eigenvectors $\varepsilon^*(A, \mathcal{T})$ are the corresponding eigenvectors of λ^* . The \mathcal{T} -dominant eigencone $\eta(A, \mathcal{T})$ is given by the intersection \mathcal{T} and the \mathcal{T} -dominant eigenvectors such that $\eta(A, \mathcal{T}) = \mathcal{T} \cap \varepsilon^*(A, \mathcal{T})$.

Definition 9 Let $P \in \mathbb{R}^{p \times n}$, where $p \leq n$, be a projection matrix. The set of P -dominant eigenvectors $\mathcal{W}(A, P)$ is given by the set $\mathcal{W}(A, P) := \{\varepsilon \in \mathcal{E}(A) \mid \exists \mathbf{v} \in \mathbb{R}^p \text{ s.t. } \varepsilon = \varepsilon^*(A, \mathcal{P}(P, \mathbf{v}))\}$, where $\mathcal{P}(P, \mathbf{v})$ is the smallest positively invariant cone containing $P^T \mathbf{v}$ and ε^* is as given in Definition 8.

Remark 3 For a given positively invariant cone \mathcal{T} , the \mathcal{T} -dominant eigenvectors correspond to the eigenvectors with largest corresponding eigenvalue that has a non-zero intersection of its operating subspace with \mathcal{T} . The P -dominant eigenvectors are then the set of all possible \mathcal{T} -dominant eigenvectors where $\mathcal{T} \subseteq \text{Im}(P^T)$. \circ

Definition 10 A matrix $H \in \mathbb{R}^{q \times q}$ is Metzler if its off-diagonal terms are non-negative.

A cone $\mathcal{X} = \text{cone}(G)$ is positive invariant if it satisfies the following result from [21, Proposition 2.1].

Proposition 1 $\text{cone}(G)$ is positively invariant for the unforced continuous LTI system (1) iff \exists a Metzler matrix $H \in \mathbb{R}^{q \times q}$ such that $GA = HG$.

3 Problem Formulation

Let a continuous LTI system be described by the n dimensional state $\mathbf{x} \in \mathbb{R}^n$, m dimensional input $\mathbf{u} \in \mathbb{R}^m$, p dimensional output $\mathbf{y} \in \mathbb{R}^p$ and system dynamics

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (2)$$

$$\mathbf{y} = C\mathbf{x}, \quad (3)$$

where A , B and C are matrices with appropriate dimensions.

The controllability property of (2) identifies if the input can be used to drive the state from any initial state to any final state. Formally *controllability* is defined as follows:

Definition 11 A continuous LTI system (2) is controllable if \exists a finite time τ , such that $\forall \mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n$ and $T \geq \tau$, \exists an input trajectory $\mathbf{u}(\cdot)$ such that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(T) = \mathbf{x}_f$ [3].

Remark 4 In addition to controllability, the properties of reachability and null controllability are defined in a similar manner with \mathbf{x}_0 fixed at $\mathbf{x}_0 = \mathbf{0}$ and \mathbf{x}_f fixed at $\mathbf{x}_f = \mathbf{0}$, respectively. For continuous LTI systems, these properties are equivalent to controllability [2]. \circ

The property of controllability has been extended to positive controllability, i.e. controllability subject to the non-negative input constraint $\mathbf{u}(\cdot) \in \mathcal{U}_\tau^+$ [12], where $\mathcal{U}_\tau^+ = \{\mathbf{u} \mid \mathbf{u} : [0, \tau] \rightarrow \mathbb{R}_{\geq 0}^m\}$, as well as the controllability of the output space \mathbb{R}^p [16], respectively, resulting in the following definitions:

Definition 12 A continuous LTI system (2) is positive controllable (PC) if \exists a control input trajectory $\mathbf{u}(\cdot) \in \mathcal{U}_T^+$ that satisfies Definition 11.

Definition 13 A continuous LTI system (2) with output (3) is output controllable if \exists a finite time τ , such that $\forall \mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{y}_f \in \mathbb{R}^p$ and $T \geq \tau$, \exists an input trajectory $\mathbf{u}(\cdot)$ defined on $t \in [0, T]$ such that $\mathbf{y}(0) = C\mathbf{x}_0$ and $\mathbf{y}(T) = \mathbf{y}_f$.

From [10, Theorem 1.4], the following proposition gives necessary and sufficient conditions for the positive controllability of the continuous LTI system (2).

Proposition 2 The continuous LTI system (2) is PC iff the following equivalent conditions hold

- There is no non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\langle \mathbf{v}, e^{At}B\mathbf{u} \rangle \leq 0$, $\forall t > 0$ and $\mathbf{u} \geq \mathbf{0}$.

- There is no non-zero eigenvector \mathbf{v} of A^T such that $\langle \mathbf{v}, B\mathbf{u} \rangle \leq 0$, $\forall \mathbf{u} \geq \mathbf{0}$ and $\text{rank}(C(A, B)) = n$, where $C(A, B) = [B \ AB \ \dots \ A^{n-1}B]$ is the controllability matrix.

Remark 5 An LTI system is controllable iff the matrix C has rank n [2]. Positive controllability is therefore a stricter property since it requires controllability and that there is no non-zero eigenvector \mathbf{v} of A^T such that $\langle \mathbf{v}, B\mathbf{u} \rangle \leq 0$, $\forall \mathbf{u} \geq \mathbf{0}$. In contrast, output controllability is typically less strict than controllability when $p < n$. Consideration of positive output controllability therefore provides a means of determining possible lower dimensional behaviours given the non-negative constraint.

The following proposition from [10, Lemma 2.4] can be used to express the inner product $\langle \mathbf{v}, e^{At}B\mathbf{u} \rangle$ in terms of the eigenvalues of A :

Proposition 3 Let $\mathbf{x}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, then $\langle \mathbf{v}, e^{At}B\mathbf{u} \rangle$ can be expanded as

$$\begin{aligned} \langle \mathbf{v}, e^{At}B\mathbf{u} \rangle &= \sum_{i=1}^k t^{j_i} e^{\lambda_i t} (\langle \mathbf{z}_i, B\mathbf{u} \rangle + \langle \boldsymbol{\eta}_i(t), B\mathbf{u} \rangle) \\ &\quad + \sum_{i=k+1}^{k+p} e^{\rho_i t} \sum_{j=1}^m t^{r_{ij}} (a_{ij}(t) + g_{ij}(t)) u_j, \quad (4) \end{aligned}$$

where $\lambda_1, \dots, \lambda_k \in \Lambda^r(A)$, $\rho_1, \dots, \rho_l \in R^c(A)$, the j_i, r_{ij} are non-negative integers, the functions $g_{ij}(t), \boldsymbol{\eta}_i(t) : \mathbb{R} \rightarrow \mathbb{R}^n$, vanish as $t \rightarrow \infty$ and the functions $a_{ij}(t)$ are sinusoids. The inner product (4) also satisfies the following conditions:

- If $\mathbf{z}_i = \mathbf{0}$, then $\boldsymbol{\eta}_i(t) = \mathbf{0}$.
- If $a_{ij}(t) = 0$, then $g_{ij}(t) = 0$.
- If $\text{rank } C(A, B) = n$, then $\forall \mathbf{v} \neq \mathbf{0} \exists$ an input such that $\langle \mathbf{v}, e^{At}B\mathbf{u} \rangle \neq 0$.

Remark 6 From Proposition 3 it can be seen that the inner product (4) used within the first equivalent condition of Proposition 2 corresponds to a sum of eigenvalue related terms for which the solution of (4) will be dominated by the dominant eigenvalue as $t \rightarrow \infty$. \circ

Consistent with Definitions 12 and 13, *positive output controllability* is defined as follows:

Definition 14 A continuous LTI system (2) and (3) is positive output controllable (POC) if \exists a control input trajectory $\mathbf{u}(\cdot) \in \mathcal{U}_T^+$ that satisfies Definition 13.

Remark 7 Consistent with Remark 4, reachability and null controllability analogies can be considered for the controllability properties of Definitions 12, 13 and 14. \circ

4 Main Results

4.1 Necessary and Sufficient Conditions for Positive Output Controllability of Continuous LTI systems

To determine necessary and sufficient conditions for the positive output controllability of the system (2) with output (3), the following proposition is first considered.

Proposition 4 *A continuous LTI system (2) with output (3) is POC iff \exists a finite time $T \geq 0$ such that it is positive output reachable.*

PROOF. By Definition 14 a system is POC if \exists a finite time $T_1 \geq 0$ such that for any $\mathbf{y}_0, \mathbf{y}_f \in \mathbb{R}^p$, a non-negative control input can be defined to drive the output from \mathbf{y}_0 to \mathbf{y}_f . Setting $\mathbf{y}_0 = \mathbf{0}$, it can be seen that the POC system must therefore be positive output reachable for time $T = T_1$.

To show sufficiency, let the output trajectory of a linear system be given by $\mathbf{y}(t) = \mathbf{y}(t, \mathbf{x}_0, \mathbf{u}(\cdot))$, where $\mathbf{u}(\cdot) \in \mathcal{U}_t^+$. At time t , the positive output reachable set $\mathcal{R}_O(t)$ of the system (2), (3) can be defined as $\mathcal{R}_O(t) = \{\mathbf{y} \in \mathbb{R}^p : \exists \mathbf{u}(\cdot) \in \mathcal{U}_t^+ \text{ s.t. } \mathbf{y} = \mathbf{y}(t, \mathbf{0}, \mathbf{u}(\cdot))\}$. If the system is positive output reachable at time $T \geq 0$, then by the definition of the reachable set $\mathcal{R}_O(t) = \mathbb{R}^p$ for all $t \geq T$.

Given an initial state $\mathbf{y}_0 \in \mathbb{R}^p$ and a corresponding $\mathbf{x}_0 \in \mathbb{R}^n$, at time T the output of a linear system (2), (3) is given by $\mathbf{y}(T) = Ce^{AT}\mathbf{x}_0 + \int_0^T Ce^{A(T-\tau)}B\mathbf{u}(\tau)d\tau$. Since T is finite, $Ce^{AT}\mathbf{x}_0$ is finite. Utilising this result and the positive output reachability of (2),(3), \exists an input trajectory $\mathbf{u}_1(\cdot) \in \mathcal{U}_T^+$ such that $\mathbf{y}_f = \int_0^T Ce^{A(T-\tau)}B\mathbf{u}_1(\tau)d\tau$ as well as an input trajectory $\mathbf{u}_2(\cdot) \in \mathcal{U}_T^+$ such that $-Ce^{AT}\mathbf{x}_0 = \int_0^T Ce^{A(T-\tau)}B\mathbf{u}_2(\tau)d\tau$. By the superposition property of linear systems it can therefore be seen that the input $(\mathbf{u}_1 + \mathbf{u}_2)(\cdot) \in \mathcal{U}_T^+$ is such that $\mathbf{y}(T) = \mathbf{y}_f$ and $\mathbf{y}(0) = \mathbf{y}_0$. This completes the proof. \square

The following theorem is obtained from Proposition 4.

Theorem 1 *A continuous LTI system is POC iff there is no non-zero vector $\mathbf{v} \in \mathbb{R}^p$ such that*

$$\langle \mathbf{v}, Ce^{At}B\mathbf{u} \rangle \leq 0, \quad \forall t > 0, \quad \forall \mathbf{u} \geq \mathbf{0}. \quad (5)$$

PROOF. Using Proposition 4, the continuous LTI system (2), (3) is POC iff it is positive output reachable. A continuous LTI system is therefore POC iff \exists a finite $T \geq 0$ such that $\mathcal{R}_O(T) = \mathbb{R}^p$.

By the definition of $\mathcal{R}_O(t)$, $\forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^m \geq \mathbf{0}$ and $\alpha \geq 0$, $\alpha\mathbf{u}_1 \in \mathbb{R}^m \geq \mathbf{0}$ and $\mathbf{u}_1 + \mathbf{u}_2 \in \mathbb{R}^m \geq \mathbf{0}$. Accordingly, by the pointwise positive scaling and addition of input sequences, it can be seen that if $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}_O(t)$, then

$\alpha\mathbf{y}_1 \in \mathcal{R}_O(t)$ and $\mathbf{y}_1 + \mathbf{y}_2 \in \mathbb{R}^p$. By Definition 1 the set $\mathcal{R}_O(t)$ is therefore always a convex cone.

Since $\mathcal{R}_O(t)$ is a convex cone, the origin lies in the interior of $\mathcal{R}_O(t)$ iff $\mathcal{R}_O(t) = \mathbb{R}^p$. By the separating hyperplane theorem [19, Theorem 2 of Section 5.12], this means that there is no non-zero vector $\mathbf{v} \in \mathbb{R}^p$ such that

$$\left\langle \mathbf{v}, \int_0^t Ce^{A\tau}B\mathbf{u}(\tau)d\tau \right\rangle \leq 0, \quad \forall t \geq 0, \quad \mathbf{u}(\cdot) \in \mathcal{U}_t^+. \quad (6)$$

The final result then follows by continuity and a special choice of $\mathbf{u}(\cdot)$ as is presented in [10, Theorem 1.4]. \square

Remark 8 *This result extends [10, Theorem 1.4] to consider the output space. An alternative interpretation of this result is that $\forall \mathbf{v} \in \mathbb{R}^p$, \exists a time $t > 0$ and an input $\mathbf{u} \geq \mathbf{0}$ such that $\langle \mathbf{v}, Ce^{At}B\mathbf{u} \rangle > 0$.* \circ

Remark 9 *For the linear system (2) with output of the non-causal form $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$, where $D \in \mathbb{R}^{p \times m}$, it can be shown using the same procedure as the proof of Theorem 1 that the continuous LTI system is POC iff there is no non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\langle \mathbf{v}, (Ce^{At}B + D)\mathbf{u} \rangle \leq 0, \quad \forall t > 0, \quad \forall \mathbf{u} \geq \mathbf{0}$. Since D is constant, its addition results in it being more likely that a linear system is POC.*

Remark 10 *Let $\mathbb{B} = \text{span}_+(B)$. Rearranging (5)¹ to the form*

$$\left\langle e^{A^T t}C^T\mathbf{v}, B\mathbf{u} \right\rangle \leq 0, \quad (7)$$

it can be seen that Theorem 1 states that the continuous LTI system (2), (3) is POC iff there is no vector $\mathbf{s}_0 \in \text{Im}(C^T)$ with dynamics described by

$$\dot{\mathbf{s}} = A^T\mathbf{s}, \quad \mathbf{s}(0) = \mathbf{s}_0, \quad (8)$$

such that $\mathbf{s}(t) \in \mathbb{B}^- \forall t > 0$. \circ

Taking the interpretation of Remark 10, the following theorem is obtained.

Theorem 2 *A continuous LTI system is POC iff there is no matrix $G \in \mathbb{R}^{q \times n}$ such that*

- $GA^T = HG$ where H is a Metzler matrix.
- $\exists \mathbf{v} \in \mathbb{R}^p$ such that $GC^T\mathbf{v} \leq \mathbf{0}$.
- $\text{cone}(G) \subseteq \mathbb{B}^-$.

PROOF. Assume that \exists a matrix $G \in \mathbb{R}^{q \times n}$ that satisfies the given conditions. Then by Proposition 1, the first condition implies that the cone described by G is positive invariant for the dynamics (8). By the second and third conditions this cone contains a possible initial

¹ By the definition of the inner product $\langle \mathbf{v}, Ce^{At}B\mathbf{u} \rangle = \mathbf{u}^T B^T e^{A^T t} C^T \mathbf{v} = \langle e^{A^T t} C^T \mathbf{v}, B\mathbf{u} \rangle$.

state $\mathbf{s}_0 \in \text{Im}(C^T)$ and is a subset of the negative polar cone \mathbb{B}^- . This means that \exists an initial state \mathbf{s}_0 for which $\mathbf{s}(t) \in \mathbb{B}^-, \forall t > 0$. As a result the continuous LTI system is not POC by Theorem 1.

Assume that (2), (3) is not POC. Then by the interpretation of Remark 10, \exists an initial state $\mathbf{s}_0 \in \text{Im}(C^T)$ such that $\mathbf{s}(t)$ given by (8) lies in the negative polar cone $\mathbb{B}^- \forall t > 0$. Let \mathcal{Y} be a positively invariant set in \mathbb{B}^- . Then by positive scaling and component addition if $\mathbf{x}, \mathbf{y} \in \mathcal{Y} \subseteq \mathbb{B}^-$, then $\mathbf{x} + \mathbf{y} \in \mathcal{Y}$. Accordingly $\mathcal{Y} \in \mathbb{B}^-$ iff there is a positively invariant cone within \mathbb{B}^- . By Proposition 1 this means that \exists a matrix $G \in \mathbb{R}^{q \times n}$ and a Metzler matrix $H \in \mathbb{R}^{q \times q}$ such that $GA^T = HG$. \square

Remark 11 *The results of this section have been presented to consider the constraints of positive controls. In a similar manner to [10], the results can be extended to apply to the more general class of constraints $\mathbf{u} \in \text{cone}(F)$ where $F \in \mathbb{R}^{q \times m}$ by replacing the set \mathbb{B}^- with the cone $\mathbb{B}^\dagger = \{\mathbf{v} \in \mathbb{R}^n \mid \langle \mathbf{v}, B\mathbf{u} \rangle \leq 0, \forall -F\mathbf{u} \geq \mathbf{0}\}$.* \circ

Remark 12 *In a manner consistent with [10, Corollary 3.11], the nonlinear time invariant system given by*

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = h(\mathbf{x}, \mathbf{u}), \quad (9)$$

where $f: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ and $h: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^p$, is locally positive output reachable if its linearisation is POC. \circ

4.2 Sufficient Conditions

From Remark 10 the vector $\mathbf{s}(t)$ with dynamics (8) must not remain in \mathbb{B}^- for all $t > 0$ if the continuous LTI system is POC. Instead of considering all $t > 0$, to simplify the verification of POC, two special cases are considered: 1) t is small, 2) t is large. A sufficient condition is provided in Theorem 3.

Theorem 3 *The linear system (2), (3) is POC if there is no vector $\mathbf{v} \in \mathbb{R}^p$ such that*

$$\langle \mathbf{v}, C\mathbf{B}\mathbf{u} \rangle \leq 0, \quad \forall \mathbf{u} \geq \mathbf{0}. \quad (10)$$

PROOF. Since $e^{A \cdot 0} = I$, condition (10) means that $\exists \mathbf{u} \geq \mathbf{0}$ such that $\langle \mathbf{v}, Ce^{At}B\mathbf{u} \rangle > 0$ at $t = 0$. By the continuity of e^{At} , (5) is continuous in t and hence the proof is completed by applying Theorem 1 at the small time $t = \epsilon > 0$. \square

Remark 13 *Theorem 3 is a separating hyperplane condition [19] on the positive span of CB . Since the positive span of a matrix is a convex cone, a separating hyperplane will not exist only if CB is positive spanning. A matrix can only be positive spanning if it has more columns than rows. This means that the number of inputs (actuators) must be redundant with respect to the system output such that $m \geq p$. Another interpretation of Theorem 3 is that*

the over-actuated continuous LTI system subject to non-negative input is equivalent in the output space to a fully actuated continuous LTI system with $\mathbf{u} \in \mathbb{R}^n$. \circ

Theorem 3 considers positive output controllability based upon the possible initial states of $\mathbf{s}_0 \in \text{Im}(C^T)$. When $\text{Im}(C^T) \cap \mathbb{B}^-$ contains only the boundary of \mathbb{B}^- , the following corollary can be applied.

Corollary 1 *Assume that for a continuous LTI system (2),(3) there exists a vector $\mathbf{v} \in \mathbb{R}^p$ such that (10) holds. The system (2),(3) is still POC if there exists a control input $\mathbf{u} \geq \mathbf{0}$ satisfying*

- $\langle \mathbf{v}, C\mathbf{B}\mathbf{u} \rangle = 0$, and
- $\langle \mathbf{v}, C\mathbf{A}\mathbf{B}\mathbf{u} \rangle > 0$.

PROOF. At time $t = 0$ there exists a $\mathbf{u} \geq \mathbf{0}$ such that the inner product (5) is equal to 0. The derivative of (5) is given by

$$\frac{d}{dt} (\langle \mathbf{v}, Ce^{At}B\mathbf{u} \rangle) |_{t=0} = \langle \mathbf{v}, C\mathbf{A}\mathbf{B}\mathbf{u} \rangle \quad (11)$$

Since $\exists \mathbf{u} \geq \mathbf{0}$ such that (5) is equal to 0 and (11) is positive at $t = 0$, then by the continuity of inner product, (5) must be positive for some small time $t = \epsilon > 0$. By Theorem 1, the system is therefore POC. \square

Remark 14 *By the interpretation of Remark 10, Theorem 3 and Corollary 1 state that for all $\mathbf{s}_0 \in \text{Im}(C^T)$, either $\mathbf{s}_0 \notin \mathbb{B}^-$ or \mathbf{s}_0 is on the boundary of \mathbb{B}^- and $\dot{\mathbf{s}}_0$ is away from the negative polar cone, respectively.* \circ

Remark 15 *Applying the interpretation of Remark 10 to Proposition 2 it can be noted that positive controllability requires that there is either a real eigenvector in \mathbb{B}^- or a complex eigenvector that is in the kernel of B^T .* \circ

Utilising Remark 15, the following theorem provides a sufficient condition for positive output controllability by looking at the behaviour of $\mathbf{s}(t)$ as $t \rightarrow \infty$.

Theorem 4 *The linear system (2), (3) is POC if there is no C -dominant eigenvector $\mathbf{w} \in \mathcal{W}(A^T, C)$ such that*

- If \mathbf{w} is complex, then $\langle \mathbf{w}, B\mathbf{u} \rangle = 0, \quad \forall \mathbf{u} \geq \mathbf{0}$.
- If \mathbf{w} is real, then $\langle \mathbf{w}, B\mathbf{u} \rangle \leq 0, \quad \forall \mathbf{u} \geq \mathbf{0}$.

PROOF. This proof consists of two steps. The first step will utilise Proposition 3 to expand the inner product (5) into terms which depend on eigenvectors and vanishing functions. The second step will then consider all possible eigenvalues of the linear system (2) and show that if the system is not POC then the dominant eigenvector \mathbf{w} must not satisfy the given conditions.

Using Proposition 3, for all $\mathbf{v} \in \mathbb{R}^p$ the inner product (5) can be expanded as

$$\begin{aligned} \langle \mathbf{s}, e^{At} B\mathbf{u} \rangle &= \sum_{i=1}^k t^{j_i} e^{\lambda_i t} (\langle \mathbf{w}_i, B\mathbf{u} \rangle + \langle \boldsymbol{\eta}_i(t), B\mathbf{u} \rangle) \\ &+ \sum_{i=k+1}^{k+l} e^{\rho_i t} \sum_{j=1}^m t^{r_{ij}} (a_{ij}(t) + g_{ij}(t)) u_j, \end{aligned} \quad (12)$$

where $\mathbf{s} = C^T \mathbf{v}$. To evaluate behaviour of this inner product as $t \rightarrow \infty$ the three possible cases according to the nomenclature of Section 2 are considered:

- (1) $\lambda_1 > \rho_1$ or ($\lambda_1 = \rho_1$ and $j_1 > \max_j(r_{1j})$).
- (2) $\lambda_1 < \rho_1$ or ($\lambda_1 = \rho_1$ and $j_1 < \max_j(r_{1j})$).
- (3) $\lambda_1 = \rho_1$ and $j_1 = \max_j(r_{1j})$.

For case 1 the real eigenvalue λ_1 is dominant. Dividing both sides of (12) by $t^{j_1} e^{\lambda_1 t}$ results in

$$\langle \mathbf{w}_1, B\mathbf{u} \rangle + J_a(t) \leq 0, \quad \forall t > 0, \mathbf{u} \geq 0 \quad (13)$$

where $J_a(t)$ is a vanishing function. As a result $\langle \mathbf{w}_1, B\mathbf{u} \rangle \leq 0, \forall \mathbf{u} \geq 0$.

For case 2 the complex eigenvalues associated with ρ_1 are dominant. Dividing both sides of (12) by $t^{r_1} e^{\rho_1 t}$, where $r_1 = \max_j(r_{ij})$ results in

$$\sum_{l_j} (a_{il_j}(t) + g_{il_j}(t)) u_{l_j} + J_b(t) \leq 0, \quad (14)$$

where $J_b(t)$ is another vanishing function. This summation is a sum of sinusoids such that it has zero mean. This means that $\sum_{l_j} (a_{il_j}(t) + g_{il_j}(t)) u_{l_j} = 0$ such that $\langle \mathbf{w}_1, B\mathbf{u} \rangle = 0$.

For case 3 the real and complex eigenvalues are equally dominant. Dividing both sides of (12) by $t^{j_1} e^{\lambda_1 t} = t^{r_1} e^{\rho_1 t}$ results in

$$\langle \mathbf{w}_1, B\mathbf{u} \rangle + \sum_{l_j} (a_{il_j}(t) + g_{il_j}(t)) u_{l_j} + J_c(t) \leq 0 \quad (15)$$

where $J_c(t)$ vanishes as $t \rightarrow \infty$. Here the sum of the sinusoids must have zero mean such that $\langle \mathbf{w}_1, B\mathbf{u} \rangle \leq 0$. Since all possible dominant eigenvalues have been considered the proof is therefore complete. \square

Remark 16 *The proof of Theorem 4 follows the structure of the proof of sufficiency for [10, Theorem 1.4]. The theorem shows that a system is POC if there is no C-dominant eigenvector that lies in \mathbb{B}^- . This differs from positive controllability [10, Theorem 1.4] whereby no eigenvector can be in \mathbb{B}^- . As a result, if $\text{rank}(C) = p$, then positive output controllability is less strict than positive controllability. \circ*

Remark 17 *Evaluation of Theorem 4 is performed by transforming the subspace defined by $\text{Im}(C^T)$ into the coordinates of the eigenvectors. This transformation can be achieved using the generalised eigenvectors of A^T as a transformation matrix V . After the transformation, C-dominant eigenvectors can be determined. Positive spanning tests can then be used to determine whether there is a ray of the C-dominant eigenvector that is in \mathbb{B}^- . \circ*

Combining Theorems 3 and 4 the following is obtained.

Corollary 2 *A continuous LTI system is POC if there is no vector $\mathbf{v} \in \mathbb{R}^p$ such that*

- $\langle \mathbf{v}, CB\mathbf{u} \rangle \leq 0, \quad \forall \mathbf{u} \geq 0$.
- *there exists a $\mathbf{v}^T C$ -dominant eigenvector $\mathbf{w} \in \mathcal{W}(A^T, \mathbf{v}^T C)$ such that*
 - *If \mathbf{w} is complex, then $\langle \mathbf{w}, B\mathbf{u} \rangle = 0, \quad \forall \mathbf{u} \geq 0$.*
 - *If \mathbf{w} is real, then $\langle \mathbf{w}, B\mathbf{u} \rangle \leq 0, \quad \forall \mathbf{u} \geq 0$ and there exists a ray \mathbf{z} of \mathbf{w} such that $\mathbf{z} \in \eta(A, \mathcal{P}(C, \mathbf{v}))$, where η is defined by Definition 8 and $\mathcal{P}(C, \mathbf{v})$ is the smallest positively invariant cone containing $C^T \mathbf{v}$.*

PROOF. This corollary requires that there is no common \mathbf{v} for which Theorems 3 and 4 are false. Since Theorem 3 considers small time and Theorem 4 consider $t \rightarrow \infty$ then if there is no common \mathbf{v} , then $\mathbf{s}(t)$ with dynamics (8) either starts outside of \mathbb{B}^- or leaves \mathbb{B}^- as $t \rightarrow \infty$. As a result the system is POC by Theorem 1. \square

Remark 18 *Theorems 3 and 4 consider the behaviour of the inner product where t is small and large, respectively. Neither condition considers the behaviour of the inner product at other times such that they cannot provide necessary and sufficient conditions for positive output controllability with the exception of some special cases. \circ*

4.3 Necessary and Sufficient Conditions: Special Cases

Theorem 5 *When $n = 2$ and $p = 1$, a continuous LTI system is POC iff it satisfies Corollary 2.*

PROOF. The sufficiency of this result is given by the proofs of Theorems 3 and 4 and Corollary 2.

For necessity assume that Corollary 2 does not hold. Then \exists a $\mathbf{v} \in \mathbb{R}^p$ such that $C^T \mathbf{v} \in \mathbb{B}^-$ and the convergent eigenvector is also in \mathbb{B}^- . Under this assumption the proof will consider the three possible sets of eigenvalue solutions: complex conjugate eigenvalues, real and distinct eigenvalues and real and repeated eigenvalues.

Let the eigenvalues be complex conjugate. Then $\mathbf{w} = \mathbf{a} \pm \mathbf{b}j \in \ker(B^T)$. Since $n = 2$, this is only the case if $\ker(B^T) = \mathbb{R}^2$. As a result, $\mathbf{z}(t)$ with linear dynamics (8) is always in \mathbb{B}^- such that by Theorem 1 the continuous LTI system must not be POC.

Let the eigenvalues be real. If A^T has a two dimensional

eigenspace or if $\mathbf{w} \in \text{Im}(C^T)$, then \mathbf{w} is positively invariant and the system must not be POC. Otherwise, let $C^T = \mathbf{c}$ be an arbitrary vector in \mathbb{R}^2 and the matrix $G = [\kappa \mathbf{c}^\perp - \kappa \mathbf{w}^\perp]^T$, where $\kappa \in \{-1, 1\}$ is set such that $\text{cone}(G) \subseteq \mathbb{B}^-$. This means that $\text{cone}(G)$ is a convex cone with extreme rays given by the rays of \mathbf{c} and \mathbf{w} that are within \mathbb{B}^- and therefore $\exists \mathbf{v} \in \mathbb{R}^p$ such that $G C^T \mathbf{v} \leq \mathbf{0}$. By Theorem 2 the system is therefore not POC iff \exists a Metzler matrix $H \in \mathbb{R}^{2 \times 2}$ such that $G A^T G^{-1} = H = \{h_{i,j}\}_{i,j=1,2}$, where G is invertible by construction.

Substituting the definition for G , it can be seen that

$$G A^T G^{-1} = \frac{\kappa^2}{\det(G)} \begin{bmatrix} \mathbf{c}^\perp & -\mathbf{w}^\perp \end{bmatrix}^T A^T \begin{bmatrix} \mathbf{w} & \mathbf{c} \end{bmatrix}. \quad (16)$$

When the eigenvalues of A^T are real and distinct, \mathbf{c} can be represented in eigenvector coordinates such that $\mathbf{c} = \alpha_1 \mathbf{w} + \alpha_2 \mathbf{z}$, where \mathbf{z} is the non-dominant eigenvector. This means that the off diagonals of (16) are given by

$$\begin{aligned} h_{21} &= \frac{\lambda_1 (\mathbf{w}^\perp)^T \mathbf{w}}{\det(G)} = 0, \\ h_{12} &= \frac{(\mathbf{c}^\perp)^T A^T \mathbf{c}}{\det(G)} = \alpha_1 (\lambda_1 - \lambda_2), \end{aligned} \quad (17)$$

where $(\lambda_1 - \lambda_2) > 0$ and $\alpha_1 > 0$ by definition.

In the case where the eigenvalues of A^T are real and repeated, $\mathbf{c} = \alpha_1 \mathbf{w} + \alpha_2 \mathbf{g}$ where \mathbf{g} is a generalised eigenvector. Since the system is two dimensional the generalised eigenvector must be such that $A^T \mathbf{g} = \lambda_1 \mathbf{g} + \sigma \mathbf{w}$ where $\sigma > 0$. Substituting the generalised eigenvector into (16) in place of \mathbf{z} it can be seen that $h_{21} = 0$ and

$$h_{12} = \frac{(\mathbf{c}^\perp)^T A^T \mathbf{c}}{\det(G)} = \alpha_2 \sigma, \quad (18)$$

where $\alpha_2 > 0$ to ensure convergence to \mathbf{w} and $\sigma > 0$ by definition. As a result h_{12} and h_{21} are non-negative for all real eigenvalues cases and therefore the system is not POC by Theorem 2. Since all possible cases are not POC it can be seen that Corollary 2 is necessary for positive output controllability thereby completing the proof. \square

Remark 19 *Corollary 2 is in general a sufficient condition that considers the behaviour of a linear system for small and large t . When $n = 2$, Theorem 5 shows that the condition is necessary and sufficient because the complex case is equivalent to $\mathbb{B}^- = \mathbb{R}^n$ and the real case monotonically converges in phase towards the component of the C -dominant eigenvector that is contained within \mathbb{B}^- . With the exception of higher dimensional systems in which all of the eigenvectors lies in the same plane as C^T , the necessity does not hold for systems with $n > 2$. This is because the effect of other eigenvectors that lie outside of \mathbb{B}^- can result in $\mathbf{s}(t)$ leaving \mathbb{B}^- in the time interval that*

the corollary does not consider. \circ

Remark 20 *When $n = p$, Theorem 1 is equivalent to Proposition 2 if $\text{Im}(C^T) = \mathbb{R}^n$. This means that when $n = p$, a continuous LTI system is POC iff it is PC and possesses a non-singular matrix C^T .* \circ

5 Illustrative Examples

5.1 Example 1

Consider the continuous LTI system (2), (3) with $n = m = 2$, $p = 1$, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -3 \\ 0 & -1 \end{bmatrix}$ and $C = [c_1 \ c_2]$, where $c_1, c_2 \in \mathbb{R}$. This system is controllable since $\text{rank}(B) = 2$. The system is however not PC. This is because the eigenvectors of A^T , given by $\mathbf{v}_1 = [1 \ 1]^T$ and $\mathbf{v}_2 = [2 \ 1]^T$, have inner products given by $\langle \mathbf{v}_1, B\mathbf{u} \rangle = -u_1 - 4u_2$ and $\langle \mathbf{v}_2, B\mathbf{u} \rangle = -2u_1 - 7u_2$. These inner products are both non-positive for all $\mathbf{u} \geq \mathbf{0}$ such that by Proposition 2 the system is not PC.

Utilising Theorem 3, the system is POC if $CB = [c_1 \ c_2] \begin{bmatrix} -1 & -3 \\ 0 & -1 \end{bmatrix} = [-c_1 \ -3c_1 - c_2]$ is positive spanning. Since $CB \in \mathbb{R}^{1 \times 2}$, this is the case if the elements of CB contain one positive term and one negative term. This means that the system is POC if $-c_1(-3c_1 - c_2) < 0$.

By Definition 7 the operational subspaces of this system are given by $\mathcal{O}(\lambda_1) = \mathbf{v}_1$ and $\mathcal{O}(\lambda_2) = \mathbf{v}_2$. This means that the C -dominant eigenvectors of the system will be either \mathbf{v}_1 or \mathbf{v}_2 . Since the C -dominant eigenvectors always contain a component in \mathbb{B}^- , it can be seen that Theorem 4 provides no new possible C matrices.

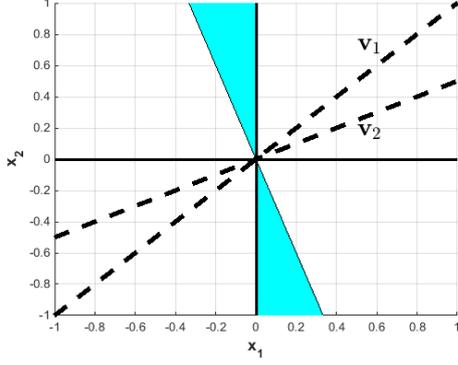
Finally by Corollary 2 it can be seen that the chosen \mathbf{v} must map to a ray of the C -dominant eigenvectors that is contained in \mathbb{B}^- . From Remark 17 this mapping can be determined utilising the transformation matrix $V = [\mathbf{v}_2 \ \mathbf{v}_1]$, such that C^T in the eigenvector coordinates is given by $C^T = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$ where $[\alpha_1 \ \alpha_2]^T = V^{-1} C^T$. This result shows that if $\alpha_1 > 0$ or if $\alpha_1 = 0$ and $\alpha_2 > 0$, then the system is POC.

Figure 1 depicts the allowable regions for the image of C^T to lie in such that the system is POC by Theorem 3 and Corollary 2. Since $C^T \in \mathbb{R}^{p \times n}$, the figure is plotted using the coordinates of the state space. It can be noted that for this case since $n = 2$, Corollary 2 identifies the complete set of possible C matrices.

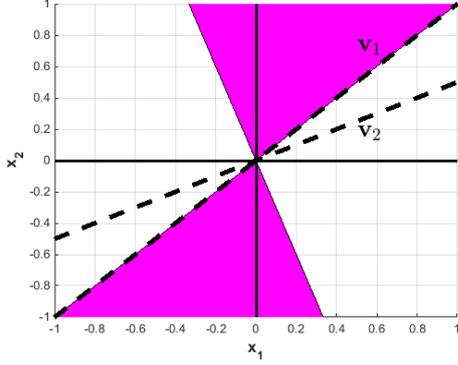
5.2 Example 2

Consider the cable driven parallel manipulator (CDPM) with system dynamics given by the compact form

$$M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + G(\mathbf{q}) + \mathbf{w}_{\text{ext}} = -L^T(\mathbf{q})\mathbf{f}, \quad (19)$$



(a) Possible C Matrices Using Theorem 3



(b) Possible C Matrices Using Corollary 2

Fig. 1. Regions where $\text{Im}(C^T)$ can be located for Positive Output Controllability

where $\mathbf{q} \in \mathbb{R}^d$ is the d -dimensional manipulator pose, $\mathbf{f} \in \mathbb{R}^m$ is the m dimensional cable force vector which satisfies the unilateral actuation constraint

$$\mathbf{f} \geq \mathbf{0}. \quad (20)$$

In equation (19), $M : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is the symmetric positive definite inertia matrix, $\mathbf{C} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$ is the Coriolis and centrifugal force vector, $\mathbf{G} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the gravitational force vector, $\mathbf{w}_{\text{ext}} \in \mathbb{R}^d$ is the external wrench vector and $L : \mathbb{R}^d \rightarrow \mathbb{R}^{m \times d}$ is the cable-joint Jacobian matrix which maps changes in the manipulator pose \mathbf{q} to changes in the cable length $\mathbf{l} \in \mathbb{R}^m$.

For CDPMs the manipulator task is typically described in terms of a p dimensional end effector pose $\mathbf{y} \in \mathbb{R}^p$ which can be expressed using the forward kinematics

$$\mathbf{y} = h(\mathbf{q}). \quad (21)$$

Consider the 2 link CDPM shown in Figure 2, where $d = 2$. Let the two rigid links be identical with a uniform distribution of mass $m_1 = m_2 = 1\text{kg}$ over the length $\ell_1 = \ell_2 = 1\text{m}$. Furthermore let the i th cable have cable mountings described by the base attachment vector

$\mathbf{r}_{O\mathbf{A}_i}$, which is a vector from the base point O to the attachment location A_i , and the rigid link attachment vector $\mathbf{r}_{G_j\mathbf{B}_i}$, where G_j refers to the centre of gravity of the j th rigid link and B_i refers to the attachment point on rigid link j for cable i . Table 1 summarises the attachment information for the CDPM.

Cable Attachment Information			
$\mathbf{r}_{O\mathbf{A}_1}$	$[-0.5 \ 0]^T$	$\mathbf{r}_{G_1\mathbf{B}_1}$	$[-0.1 \ 0.3]^T$
$\mathbf{r}_{O\mathbf{A}_2}$	$[-1.5 \ 0]^T$	$\mathbf{r}_{G_2\mathbf{B}_2}$	$[-0.1 \ 0.3]^T$
$\mathbf{r}_{O\mathbf{A}_3}$	$[0.5 \ 0]^T$	$\mathbf{r}_{G_1\mathbf{B}_3}$	$[0.1 \ 0.3]^T$
$\mathbf{r}_{O\mathbf{A}_4}$	$[1.5 \ 0]^T$	$\mathbf{r}_{G_2\mathbf{B}_4}$	$[0.1 \ 0.3]^T$

Table 1
Cable Attachment Information

Assume that the manipulator has no external disturbances and lies in the horizontal plane such that $\mathbf{w}_{\text{ext}} = \mathbf{0}$ and $G(\mathbf{q}) = \mathbf{0}$. The model dynamics can therefore be written in the form of (19) subject to the constraint equation (20), where for the choice of mechanism pose $\mathbf{q} = [q_1 \ q_2]^T$ with q_1 and q_2 as shown in Figure 2, the terms $M(\mathbf{q})$, $\mathbf{C}(\mathbf{q})$ and $L(\mathbf{q})$ are given by

$$M(\mathbf{q}) = \begin{bmatrix} \cos(q_2) + \frac{5}{3} & \frac{1}{2} \cos(q_2) + \frac{1}{3} \\ \frac{1}{2} \cos(q_2) + \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \quad (22)$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -\frac{1}{2} \dot{q}_2 \sin(q_2) (2\dot{q}_1 + \dot{q}_2) & \frac{1}{2} \dot{q}_1^2 \sin(q_2) \end{bmatrix}^T, \quad (23)$$

$$L(\mathbf{q}) = \begin{bmatrix} (\mathbf{r}_{O\mathbf{B}_1} \times \hat{\mathbf{l}}_1)^T & 0_{(1,3)} \\ 0_{(1,3)} & (\mathbf{r}_{O\mathbf{B}_2} \times \hat{\mathbf{l}}_2)^T \\ (\mathbf{r}_{O\mathbf{B}_3} \times \hat{\mathbf{l}}_3)^T & 0_{(1,3)} \\ 0_{(1,3)} & (\mathbf{r}_{O\mathbf{B}_4} \times \hat{\mathbf{l}}_4)^T \end{bmatrix} \begin{bmatrix} 0 \ 0 \\ 1 \ 0 \\ 0 \ 0 \\ 0 \ 0 \\ 1 \ 1 \\ 0 \ 0 \end{bmatrix}, \quad (24)$$

where $\hat{\mathbf{l}}_i = \widehat{\mathbf{r}_{\mathbf{A}_i\mathbf{B}_i}}$ and both $\hat{\mathbf{l}}_i$ and $\mathbf{r}_{O\mathbf{B}_i}$ are functions of the mechanism pose \mathbf{q} for $i = 1 \dots 4$. Let the manipulator task be described in terms of the vector y in Figure 2 with forward kinematics (21) where

$$h(\mathbf{q}) = \cos(q_1) + \cos(q_1 + q_2). \quad (25)$$

Defining the manipulator state $\mathbf{x} \in \mathbb{R}^4$, to be $\mathbf{x} = [\mathbf{x}_1^T \ \mathbf{x}_2^T]^T = [\mathbf{q}^T \ \dot{\mathbf{q}}^T]^T$, the system input to be $\mathbf{u} = \mathbf{f} \in \mathbb{R}^4$ and the system output to be y . The dynamics (19) of the robot can be represented in the state space form

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_2 \\ -M(\mathbf{x}_1)^{-1} (\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2) + L^T(\mathbf{x}_1)\mathbf{u}) \end{bmatrix}, \quad (26)$$

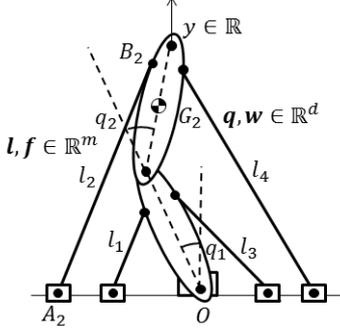


Fig. 2. Example 2 Link Cable Driven Manipulator

where M , \mathbf{C} and L are given by equations (22), (23) and (24), respectively. The system (26) is subject to the unilateral (non-negative) input constraints $\mathbf{u} \geq \mathbf{0}$ and has an output equation given by (25).

The cable robot dynamics (19) is that of a continuous nonlinear time invariant system such that linearisation can be used to evaluate the positive output controllability of the system through Remark 12. Taking the linearisation about the equilibrium triple $(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}}, \mathbf{y}^{\text{eq}}) = ([\mathbf{q}^{\text{eq}} \ \mathbf{0}]^T, \mathbf{u}^{\text{eq}}, \mathbf{y}^{\text{eq}})$ it can be seen that the nominal linearised model is given by

$$\delta \dot{\mathbf{x}} = A \delta \mathbf{x} + B \delta \mathbf{u}, \quad \delta \mathbf{y} = C \delta \mathbf{y}, \quad (27)$$

where $\delta \mathbf{x} = \mathbf{x} - \mathbf{x}^{\text{eq}}$, $\delta \mathbf{y} = \mathbf{y} - \mathbf{y}^{\text{eq}}$, $\delta \mathbf{u} = \mathbf{u} - \mathbf{u}^{\text{eq}}$ and the state, input and output matrices of the linearised system are given by

$$A = \begin{bmatrix} 0_{(2 \times 2)} & I_2 \\ \frac{\partial}{\partial \mathbf{x}_1} (-M(\mathbf{x}_1)^{-1} L^T(\mathbf{x}_1) \mathbf{u})|_{(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})} & 0_{(2 \times 2)} \end{bmatrix}, \quad B = \begin{bmatrix} 0_{(2 \times 4)} \\ -M(\mathbf{q}^{\text{eq}})^{-1} L^T(\mathbf{q}^{\text{eq}}) \end{bmatrix} \text{ and } C = \begin{bmatrix} \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}}|_{(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})} & 0_{(1 \times 2)} \end{bmatrix},$$

where M , L and h are given by (22), (24) and (25).

Let the equilibrium triple represent the nominal operating point $(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}}, \mathbf{y}^{\text{eq}}) = ([\frac{\pi}{6} \ \frac{\pi}{3} \ 0 \ 0]^T, \mathbf{0}, \frac{\sqrt{3}}{2})$. Substitution of this nominal operating point into the expressions for the state matrix, input matrix and output matrices results in $A = \begin{bmatrix} 0_{(2 \times 2)} & I_2 \\ 0_{(2 \times 2)} & 0_{(2 \times 2)} \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.436 & 0.0123 & -0.294 & -0.681 \\ -0.734 & 2.031 & 0.515 & 1.691 \end{bmatrix}$ and $C = [-1.5 \ -1 \ 0 \ 0]$.

The positive output controllability of the linearisation (27) can be evaluated using Theorem 1 whereby from the given form for the state matrix it can be seen that $e^{At} = \begin{bmatrix} I_2 & tI_2 \\ 0_{2 \times 2} & I_2 \end{bmatrix}$, such that $Ce^{At}B\mathbf{u} = t[0.109 \ -2.216 \ -0.074 \ -0.669]\mathbf{u}$.

As a result the inner product $\langle v, Ce^{At}B\mathbf{u} \rangle$ is given by

$$\langle v, Ce^{At}B\mathbf{u} \rangle = -tv[-0.109 \ 2.216 \ 0.074 \ 0.669]\mathbf{u}. \quad (28)$$

If $v > 0$, then for all time $t > 0$ the choice of input $\mathbf{u} = [1 \ 0 \ 0 \ 0]^T$ results in the inner product (28) being positive. Similarly if $v < 0$, the choice of input $\mathbf{u} = [0 \ 1 \ 0 \ 0]^T$ results in (28) being positive for all time $t > 0$. This means that there is no non-zero v such that $\langle v, Ce^{At}B\mathbf{u} \rangle$ is non-positive for all times $t > 0$ and positive input $\mathbf{u} \geq \mathbf{0}$. As a result the nominal operating point has a POC linearisation by Theorem 1.

By considering Remark 12, it can be seen that the nominal operating point is locally POC. An exact region of the state space for which positive output controllability holds is however at this point unknown. The approach of workspace analysis considered in [7] can be conducted to approximate the PC regions by looking at a sampled representation of the state space. Figure 3 depicts the workspace consisting of all sampled poses with POC linearisations. It can be seen that the sampled poses in the immediate vicinity of the nominal operating point (depicted by the red cross) are also POC. This suggests that the nominal operating point $([\frac{\pi}{6} \ \frac{\pi}{3} \ 0 \ 0]^T, \mathbf{0}, \frac{\sqrt{3}}{2})$ belongs to a POC region.

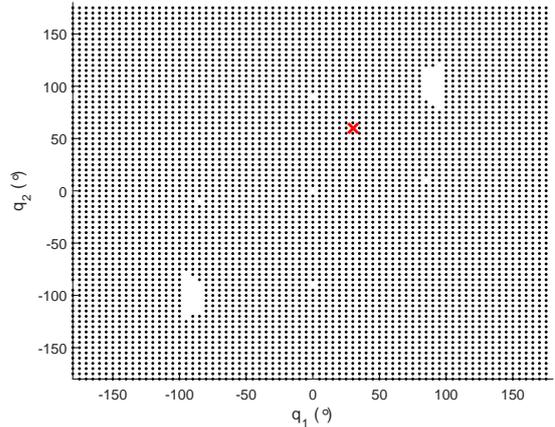


Fig. 3. Workspace of POC Linearisations

6 Conclusion

In this paper, the concept of positive output controllability was evaluated for continuous LTI systems. A set of necessary and sufficient conditions were derived for which it was noted that practical evaluation of the conditions was made difficult due to time dependence and nonlinear constraints. To address these practical issues a geometric interpretation of positive controllability conditions was provided from which sufficient conditions were identified. These conditions were then applied to numerical examples to illustrate their application and to gain an intuition of the conservativeness of the results. Future work will look to identify more computationally efficient necessary and sufficient conditions for the positive output controllability of continuous LTI systems and to extend the results to consider the non-local posi-

tive output controllability analysis of general nonlinear systems.

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