

On Positive Output Controllability and Cable Driven Parallel Manipulators

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Abstract— This paper introduces a new concept: the positive output controllability for a multi-link cable driven parallel manipulator (CDPM). This class of mechanism is characterised by the use of cable actuation which is constrained to be non-negative at all times. For simplicity, a linear-time-invariant system (LTI) is considered, which corresponds to the local behavior of the CDPMs around an particular equilibrium. A necessary and sufficient condition is provided to ensure that a LTI system is positive output controllable. The obtained results are verified by using a simulation example of a 2-link CDPM.

I. INTRODUCTION

Cable driven parallel manipulators (CDPMs) are a class of mechanism in which the actuation is provided by cables in place of rigid link actuators. The lightweight nature of the cables results in these mechanisms possessing a number of benefits such as reduced inertia and increased system reconfigurability. This has resulted in a range of applications for CDPMs including manufacturing and construction [1], [2], humanoid robots [3], [4] and the modelling of biomechanics systems [5].

Different from other robotic manipulators controlled by motors, a unique feature in the study of CDPMs is that the actuating cables can only operate under tension (*positive cable force*). This results in constraints in any control input, complicating the subsequent control and analysis of CDPMs [6], [7]. One method that is used to simplify the control of these mechanisms is to restrict the robot to operate only at those poses which satisfy a condition known as the wrench closure condition [8], [9]. This condition corresponds to a requirement that the system can produce force in all its degrees of freedom. The wrench closure condition is closely linked to the well-known concept of *local controllability* in control engineering. More specifically, wrench closure is a special case of local controllability, which shows that a desired system trajectory can be achieved by designing an appropriate control input.

Controllability plays a crucial role in control engineering. It determines whether an engineered system can generate the desired behavior, for example, driving from arbitrary initial state to the arbitrary final state in a given time interval. The analysis of the controllability properties of a dynamic system (whether it is linear or nonlinear) in state space or subset of the state space, was first introduced by Kalman

[10] and seeks to determine if a controller can be applied to generate a desired state space behaviour. Controllability has been studied for a wide range of systems, where for the case of linear time invariant (LTI) systems, a range of necessary and sufficient conditions have been identified to evaluate the system's controllability [10], [11]. Although there does not exist the necessary and sufficient condition for a general nonlinear dynamic systems, the local controllability of equilibrium states can be evaluated through sufficient conditions derived from linearisation [12] and Lie algebra [13]. The specific case of controllability subject to non-negative input constraints (positive controllability) has been widely investigated for systems such as one way valves [14] and the antivibration control of pendulum systems [15]. This analysis has resulted in necessary and sufficient conditions for LTI systems that are both continuous [15]–[18] and discrete [19], in addition to sufficient conditions for the local positive controllability of non-linear systems [16], [20].

For many CDPMs, such as multi-link cable driven manipulators [21], the desired manipulator behaviour may be described in terms of the lower dimensional end effector pose rather than the mechanism's generalised coordinates. It is therefore natural to consider the control of the end effector pose in place of the control of the mechanism state. For this reason, the wrench closure condition which requires that all accelerations in the generalised coordinates can be produced with positive cable force, represents a very strict restriction of the mechanism when compared to conditions that only consider the end effector (or output) space, such as local *output controllability* [22], [23]. In the evaluation of output controllability, necessary and sufficient conditions for LTI output controllability are well established [24]. There has however been little consideration of output controllability subject to constraints, for which there are no known results for the output controllability of a system subject to non-negative input constraints.

In this paper the property of positive output controllability is defined to consider the output controllability of systems that are subject to non-negative input constraints. Extending upon the positive controllability conditions for a continuous LTI system derived in [16], a necessary and sufficient condition for the **positive output controllability** of a continuous LTI system is derived. It is shown that through

linearisation this condition can be utilised as a sufficient condition for the local positive output controllability of a non-linear time invariant system. The condition is then applied to the problem of evaluating the positive output controllability of a multi-link cable driven mechanism about a nominal operating point (or equilibrium).

The remainder of the paper is organised as follows: In Section II, some preliminary definitions and the problem statement are presented. The main results are presented in Section III. These results are then applied to the problem of determining if nominal operating point of a multilink cable driven manipulator is positive output controllable in Section IV. Section V concludes the paper and presents areas of future work.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Preliminaries

Denote the set of real numbers as \mathbb{R} , the zero matrix with m rows and n columns as $0_{(m \times n)}$ and the square identity matrix with m rows as I_m . If the vector $\mathbf{x} = [x_1 \dots x_n]^T \in \mathbb{R}^n$ satisfies $x_i > 0$ ($x_i \geq 0$) for all $i \in \{1, \dots, n\}$, then \mathbf{x} is said to be positive (non-negative) and is denoted by $\mathbf{x} > \mathbf{0}$ ($\mathbf{x} \geq \mathbf{0}$). For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the Euclidean inner product is denoted $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{x}$ and the Euclidean norm is denoted $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}}$. A set $\mathcal{X} \subseteq \mathbb{R}^n$ is a *cone* if for all $\mathbf{x} \in \mathcal{X}$ and $\alpha \geq 0$, $\alpha \mathbf{x} \in \mathcal{X}$. A cone \mathcal{X} is a *convex cone* if for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{x} + \mathbf{y} \in \mathcal{X}$.

Let a continuous time invariant system $\Sigma(\mathbf{x}, \mathbf{u})$ be described by the n dimensional state $\mathbf{x} \in \mathbb{R}^n$, m dimensional input $\mathbf{u} \in \mathbb{R}^m$ as follows

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (1)$$

where $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$. The system Σ_L is said to be continuous linear time invariant if its dynamics are given

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (2)$$

where $A \in \mathbb{R}^{n \times n}$ is the state matrix and $B \in \mathbb{R}^{n \times m}$ is the input matrix.

The system Σ is said to satisfy the positive controllability property [18] under the following definition

Definition 1: A system $\Sigma(\mathbf{x}, \mathbf{u})$ is *positive controllable* if there exists a finite time τ , such that $\forall \mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n$ and $T \geq \tau$, there exists a non-negative input trajectory, for which $\mathbf{u}(t) \geq \mathbf{0} \forall t \geq 0$, such that $\mathbf{x}(T) = \mathbf{x}_f$ and $\mathbf{x}(0) = \mathbf{x}_0$.

Remark 1: The properties of *positive reachability* and *positive null controllability* correspond to positive controllability where the initial state is fixed as $\mathbf{x}_0 = \mathbf{0}$ and the final state is fixed at $\mathbf{x}_f = \mathbf{0}$, respectively. For continuous LTI systems, positive reachability, positive null controllability and positive controllability are equivalent system properties [16]. \circ

The state $\mathbf{x} \in \mathbb{R}^n$ is said to be locally positive controllable under the following definition

Definition 2: The state \mathbf{x}^{eq} is locally positive controllable if there exists a $r > 0$ such that for all $\bar{\mathbf{x}}_0, \bar{\mathbf{x}}_f \in \mathcal{B}(\mathbf{x}^{\text{eq}}, r)$, where $\mathcal{B}(\mathbf{x}^{\text{eq}}, r) = \{\bar{\mathbf{x}} \in \mathbb{R}^n \mid \|\bar{\mathbf{x}} - \mathbf{x}^{\text{eq}}\| \leq r\}$, there exists a non-negative input trajectory and a finite time $T \geq 0$ such that $\mathbf{x}(0) = \bar{\mathbf{x}}_0$ and $\mathbf{x}(T) = \bar{\mathbf{x}}_f$.

Remark 2: This definition of local positive controllability represents an extension of the local controllability definition in [12]. \circ

From [16, Theorem 1.4], the following proposition gives necessary and sufficient conditions for positive controllability of a continuous LTI system.

Proposition 1: The continuous LTI system $\Sigma_L(\mathbf{x}, \mathbf{u})$ is positive controllable if either of the following conditions are true

- There is no non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\langle \mathbf{v}, e^{At} B \mathbf{u} \rangle \leq 0, \quad \forall t > 0, \quad \mathbf{u} \geq \mathbf{0}$.
- There is no non-zero eigenvector \mathbf{v} of A^T such that $\langle \mathbf{v}, B \mathbf{u} \rangle \leq 0, \quad \forall \mathbf{u} \geq \mathbf{0}$ and $\text{rank}(\mathcal{C}(A, B)) = n$, where $\mathcal{C}(A, B) = [B \quad AB \quad \dots \quad A^{n-1}B]$.

Remark 3: From the second equivalent condition of Proposition 1 it can be seen that the controllability of continuous LTI systems subject to unilateral input constraints requires that the system is completely controllable and that it obeys an additional condition related to the constraint set. This means that positive controllability represents a stricter condition that captures the effect of a non-negative input constraints on the capability of a system. \circ

The following proposition from [16, Corollary 3.11] provides a relationship between the local positive controllability of the nonlinear system Σ at the equilibrium pair $(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})$ with the positive controllability of the linearisation of Σ about that point.

Proposition 2: The equilibrium pair $(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})$, where $\mathbf{u}^{\text{eq}} \geq \mathbf{0}$, is locally positive reachable if its linearisation

$$\delta \dot{\mathbf{x}} = A \delta \mathbf{x} + B \delta \mathbf{u}, \quad \delta \mathbf{x}(0) = \delta \mathbf{x}_0, \quad (3)$$

where $A = \frac{\partial f}{\partial \mathbf{x}}|_{(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})}$, $B = \frac{\partial f}{\partial \mathbf{u}}|_{(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})}$, $\delta \mathbf{x} = \mathbf{x} - \mathbf{x}^{\text{eq}}$ and $\delta \mathbf{u} = \mathbf{u} - \mathbf{u}^{\text{eq}}$, is positive controllable.

Remark 4: The state \mathbf{x}^{eq} is locally positive reachable if it satisfies the local positive controllability condition of Definition 2 with fixed initial state of \mathbf{x}^{eq} . \circ

The equilibrium pair can be found to lie within a positive controllable set under the following lemma

Lemma 1: Let $\mathcal{D} \subset \mathbb{R}^n$ be a set of connected equilibrium points. If for all $\mathbf{x} \in \mathcal{D}$, the linearised system Σ_L is positive controllable then the set \mathcal{D} forms a positive controllable set.

Proof: If all $\mathbf{x} \in \mathcal{D}$ have positive controllable linearisations then each equilibrium pair is locally positive reachable such that the neighbouring elements within \mathcal{D} can be reached. Since the elements of \mathcal{D} are connected to one another, from any initial state $\mathbf{x}_0 \in \mathcal{D}$, the desired state $\mathbf{x}_f \in \mathcal{D}$ can be reached utilising a sequence of trajectories to neighbouring points. \blacksquare

Remark 5: The equilibrium pair $(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})$ therefore lies within a positive controllable set if it has a positive controllable linearisation and is connected to another equilibrium pair with positive controllable linearisation. \circ

B. Problem Statement

Let a continuous non-linear time-invariant system $\Sigma(\mathbf{x}, \mathbf{u})$ be described by the system dynamics (1) and the p dimensional output $\mathbf{y} \in \mathbb{R}^p$, with output equation

$$\mathbf{y} = h(\mathbf{x}), \quad (4)$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$. Deduce sufficient conditions for which the nominal operating equilibrium $(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}}, \mathbf{y}^{\text{eq}})$ satisfies the property of local positive output controllability, where positive output controllability is defined as an extension of Definition 1 that considers the output space.

Definition 3: A continuous system $\Sigma(\mathbf{x}, \mathbf{u}, \mathbf{y})$ is *positive output controllable* (POC) if \exists a finite time T , such that $\forall \mathbf{y}_0, \mathbf{y}_f \in \mathbb{R}^p$ and $T \geq \tau$, \exists an input trajectory a non-negative input trajectory, for which $\mathbf{u}(t) \geq \mathbf{0} \forall t \geq 0$ such that $\mathbf{y}(T) = \mathbf{y}_f$ and $\mathbf{y}(0) = \mathbf{y}_0$.

Remark 6: In accordance with Remark 1, positive output reachability and positive output null controllability analogies can be defined for the positive output controllability property of Definitions 3 by using the output in place of the state. \circ

Remark 7: In order to simplify the presentation, the focus of our paper is to provide sufficient and necessary condition of the positive output controllability for the following continuous LTI system with the output:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u}, \\ \mathbf{y} &= C\mathbf{x}, \end{aligned} \quad (5)$$

where $C \in \mathbb{R}^{p \times n}$, will therefore be considered. \circ

III. MAIN RESULT

To determine necessary and sufficient conditions for positive output controllability of the continuous LTI system (2) with linear output (5), the following proposition is first proposed.

Proposition 3: A continuous LTI system (2) with output equation (5) is positive output controllable (POC) iff \exists a finite time $T \geq 0$ such that it is completely positive output reachable from the origin.

Proof: Necessity is determined by considering Definition 3. More precisely a system is POC if there exists a finite time $T_1 \geq 0$ such that for any $\mathbf{y}_0, \mathbf{y}_f \in \mathbb{R}^p$, a non-negative control input drive the output from \mathbf{y}_0 to \mathbf{y}_f . Setting $\mathbf{y}_0 = \mathbf{0}$, it can be seen that the system must therefore be positive output reachable for time $T = T_1$.

To show sufficiency, let the output trajectory of a linear system be given by $\mathbf{y}(t) = \mathbf{y}(t, \mathbf{x}_0, \mathbf{u}(\cdot))$, where $\mathbf{u}(\cdot) \in \mathcal{U}_\tau^+$ and $\mathcal{U}_\tau^+ = \{\mathbf{u} \mid \mathbf{u} : [0, \tau] \rightarrow \mathbb{R}_{\geq 0}^m\}$. At time t , the positive output reachable set of the continuous LTI system, $\mathcal{R}_O(t)$

can be defined as

$$\mathcal{R}_O(t) = \{\mathbf{y} \in \mathbb{R}^p : \exists \mathbf{u}(\cdot) \text{ s.t. } \mathbf{y} = \mathbf{y}(t, \mathbf{0}, \mathbf{u}(\cdot))\}. \quad (6)$$

If the system is completely positive output reachable at time $T \geq 0$, then by the definition of the reachable set $\mathcal{R}_O(t) = \mathbb{R}^p$ for all $t \geq T$.

Given an initial state $\mathbf{y}_0 \in \mathbb{R}^p$ and a corresponding $\mathbf{x}_0 \in \mathbb{R}^n$, at time T the output of a linear system (5) is given by $\mathbf{y}(T) = Ce^{AT}\mathbf{x}_0 + \int_0^T Ce^{A\tau}B\mathbf{u}(\tau)d\tau$. Since T is finite, $Ce^{AT}\mathbf{x}_0$ is finite. Utilising this result and the positive output reachability of (5), there exists an input trajectory $\mathbf{u}_1(\cdot) \in \mathcal{U}_T^+$ such that $\mathbf{y}_f = \int_0^T Ce^{A\tau}B\mathbf{u}_1(\tau)d\tau$ as well as an input trajectory $\mathbf{u}_2(\cdot) \in \mathcal{U}_T^+$ such that $-Ce^{AT} = \int_0^T Ce^{A\tau}B\mathbf{u}_2(\tau)d\tau$. By the superposition property of linear systems it can therefore be seen that the input $(\mathbf{u}_1 + \mathbf{u}_2)(\cdot) \in \mathcal{U}_T^+$ is such that $\mathbf{y}(T) = \mathbf{y}_f$ and $\mathbf{y}(0) = \mathbf{y}_0$. This completes the proof. \blacksquare

Remark 8: In the presence of input constraints it is shown in [16, Theorem 1.4] that positive reachability, positive null controllability and positive controllability are equivalent. This makes use of the negative system $\Sigma^-(\mathbf{x}, \mathbf{u})$ described by the dynamics

$$\dot{\mathbf{x}} = -A\mathbf{x} - B\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0. \quad (7)$$

This definition does not consider the output of a continuous LTI system. As a result an alternative proof method has therefore been utilised in this paper. \circ

The following theorem is obtained to provide necessary and sufficient conditions for the positive output controllability of the linear system (2) with output (5) from Proposition 3.

Theorem 1: A Continuous LTI system is POC iff there is no non-zero vector $\mathbf{v} \in \mathbb{R}^p$ such that

$$\langle \mathbf{v}, Ce^{At}B\mathbf{u} \rangle \leq 0, \quad \forall t > 0, \quad \forall \mathbf{u} \geq \mathbf{0}. \quad (8)$$

Proof: Using Proposition 3, the continuous LTI system (2) with output (5) is POC iff it is positive output reachable. A continuous LTI system is therefore POC iff \exists a finite $T \geq 0$ such that $\mathcal{R}_O(T) = \mathbb{R}^p$.

By the definition of $\mathcal{R}_O(t)$, $\forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}_{\geq 0}^m$ and $\alpha \geq 0$, $\alpha\mathbf{u}_1 \in \mathbb{R}_{\geq 0}^m$ and $\mathbf{u}_1 + \mathbf{u}_2 \in \mathbb{R}_{\geq 0}^m$. Accordingly, by the pointwise positive scaling and addition of input sequences, it can be seen that if $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{R}_O(t)$, then $\alpha\mathbf{y}_1 \in \mathcal{R}_O(t)$ and $\mathbf{y}_1 + \mathbf{y}_2 \in \mathbb{R}^p$. Using the preliminary definitions of Section II-A it can therefore be seen that the set $\mathcal{R}_O(t)$ is always a convex cone.

Since $\mathcal{R}_O(t)$ is a convex cone, the origin lies in the interior of $\mathcal{R}_O(t)$ iff $\mathcal{R}_O(t) = \mathbb{R}^p$. By the separating hyperplane theorem [25, Theorem 2 of Section 5.12], this means that there is no non-zero vector $\mathbf{v} \in \mathbb{R}^p$ such that

$$\left\langle \mathbf{v}, \int_0^t Ce^{A\tau}B\mathbf{u}(\tau)d\tau \right\rangle \leq 0, \quad \forall t \geq 0, \quad \mathbf{u}(\cdot) \in \mathcal{U}_t^+. \quad (9)$$

The final result then follows by continuity and a special choice of $\mathbf{u}(\cdot)$ as is presented in [16, Theorem 1.4]. ■

Remark 9: This result extends [16, Theorem 1.4] to consider the output space. An alternative interpretation of this result is that $\forall \mathbf{v} \in \mathbb{R}^p, \exists$ a time $t > 0$ and an input $\mathbf{u} \geq \mathbf{0}$ such that $\langle \mathbf{v}, Ce^{At}B\mathbf{u} \rangle > 0$. ◦

Using Proposition 2, the result of Theorem 1 can be applied to evaluate the local positive output controllability of the nonlinear system (1) with output equation (4). This results in the following proposition

Proposition 4: The equilibrium triple $(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}}, \mathbf{y}^{\text{eq}})$, where $\mathbf{u}^{\text{eq}} \geq \mathbf{0}$, is locally positive output reachable if its linearisation

$$\begin{aligned} \delta \dot{\mathbf{x}} &= A\delta \mathbf{x} + B\delta \mathbf{u}, & \delta \mathbf{x}(0) &= \delta \mathbf{x}_0, \\ \delta \mathbf{y} &= C\delta \mathbf{x}, \end{aligned} \quad (10)$$

where $A = \frac{\partial f}{\partial \mathbf{x}}|_{(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})}$, $B = \frac{\partial f}{\partial \mathbf{u}}|_{(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})}$, $C = \frac{\partial h}{\partial \mathbf{x}}|_{(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})}$, $\delta \mathbf{x} = \mathbf{x} - \mathbf{x}^{\text{eq}}$, $\delta \mathbf{u} = \mathbf{u} - \mathbf{u}^{\text{eq}}$ and $\delta \mathbf{y} = \mathbf{y} - \mathbf{y}^{\text{eq}}$, is positive output controllable.

Remark 10: Using Lemma 1 the local positive output reachability property of the linearisation can result in local positive output controllability provided that the equilibrium triple is connected to another equilibrium triple with positive output controllable linearisation. In this case the connected point must lie on the same output manifold such that connectivity in the output space implies connectivity in the state space. ◦

IV. POSITIVE CONTROLLABILITY OF A MULTI-LINK CABLE DRIVEN ROBOT

To demonstrate the application of Theorem 1 and Proposition 4, the positive output controllability of a multi-link cable driven manipulator about a desired operating point will be considered in this section.

A. Dynamics of a CDPM

Cable driven parallel manipulators form a class of mechanism that it is characterised by cable actuation, in which the cables can only pull and not push, in addition to system dynamics which can always be represented in the compact form

$$M(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) + \mathbf{w}_{\text{ext}} = -L^T(\mathbf{q})\mathbf{f}, \quad (11)$$

where $\mathbf{q} \in \mathbb{R}^d$ is the d -dimensional manipulator pose, $\mathbf{f} \in \mathbb{R}^m$ is the m dimensional cable force vector which satisfies the unilateral actuation constraint

$$\mathbf{f} \geq \mathbf{0}. \quad (12)$$

In equation (11), $M: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ is the mass inertia matrix, $\mathbf{C}: \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$ is the Coriolis and centrifugal force vector, $\mathbf{G}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the gravitational force vector, $\mathbf{w}_{\text{ext}} \in \mathbb{R}^d$ is the external wrench vector and $L: \mathbb{R}^d \rightarrow \mathbb{R}^{m \times d}$ is the cable-joint Jacobian matrix which maps changes in the manipulator pose \mathbf{q} to changes in the cable length $\mathbf{l} \in \mathbb{R}^m$.

For this class of mechanism, the final task is typically described in terms of a p dimensional end effector pose $\mathbf{y} \in \mathbb{R}^p$ which can be expressed in terms of the manipulator pose through the forward kinematics equation

$$\mathbf{y} = h(\mathbf{q}). \quad (13)$$

Consider the 2 link 4 cable driven mechanism shown in Figure 1, where $d = 2$ and $m = 4$. Let the two rigid links be identical with a uniform distribution of mass $m_1 = m_2 = 1\text{kg}$ over the length $\ell_1 = \ell_2 = 1\text{m}$. Furthermore let the i^{th} , $i = 1, \dots, 4$, cable have cable mountings described by the base attachment vector \mathbf{r}_{OA_i} , which is a vector from the base point O to the attachment location A_i , and the rigid link attachment vector $\mathbf{r}_{\text{G}_j\text{B}_i}$, where G_j refers to the centre of gravity of the j^{th} rigid link and B_i refers to the attachment point on rigid link j for cable i . Table I summarises the parameter information for the cable driven manipulator.

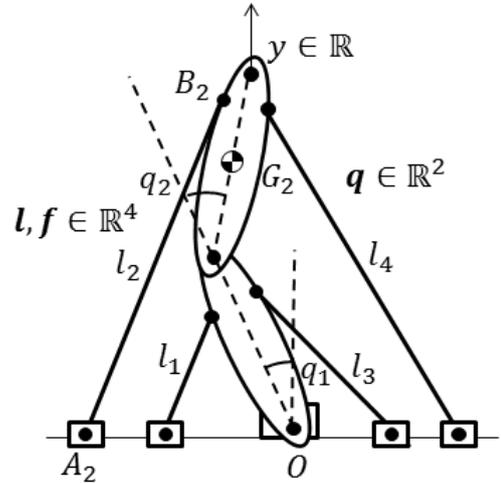


Fig. 1. Example 2 Link Cable Driven Manipulator

Cable Attachment Parameters			
\mathbf{r}_{OA_1}	$[0 \ 0.5]^T$	$\mathbf{r}_{\text{G}_1\text{B}_1}$	$[0.3 \ 0.1]^T$
\mathbf{r}_{OA_2}	$[0 \ 1.5]^T$	$\mathbf{r}_{\text{G}_2\text{B}_2}$	$[0.3 \ 0.1]^T$
\mathbf{r}_{OA_3}	$[0 \ -0.5]^T$	$\mathbf{r}_{\text{G}_1\text{B}_3}$	$[0.3 \ -0.1]^T$
\mathbf{r}_{OA_4}	$[0 \ -1.5]^T$	$\mathbf{r}_{\text{G}_2\text{B}_4}$	$[0.3 \ -0.1]^T$

TABLE I
MECHANISM PARAMETERS

Assume that the manipulator has no external disturbances and lies in the horizontal plane such that $\mathbf{w}_{\text{ext}} = \mathbf{0}$ and $\mathbf{G}(\mathbf{q}) = \mathbf{0}$, respectively. The model dynamics can therefore be written in the form of (11) subject to the constraint equation (12), where for the choice of mechanism pose $\mathbf{q} = [q_1 \ q_2]^T$ with q_1 and q_2 as shown in Figure 1, the mass matrix, Coriolis and Centrifugal vector and cable-joint

Jacobian matrix are given by

$$M(\mathbf{q}) = \begin{bmatrix} \cos(q_2) + \frac{3}{2} & \frac{1}{2} \cos(q_2) + \frac{1}{4} \\ \frac{1}{2} \cos(q_2) + \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad (14)$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -\frac{1}{2} \dot{q}_2 \sin(q_2) (2\dot{q}_1 + \dot{q}_2) \\ \frac{1}{2} \dot{q}_1^2 \sin(q_2) \end{bmatrix} \quad (15)$$

and

$$L(\mathbf{q}) = \begin{bmatrix} (\mathbf{r}_{\text{OB}_1} \times \hat{\mathbf{l}}_1)^T & 0_{(1,3)} \\ 0_{(1,3)} & (\mathbf{r}_{\text{OB}_2} \times \hat{\mathbf{l}}_2)^T \\ (\mathbf{r}_{\text{OB}_3} \times \hat{\mathbf{l}}_3)^T & 0_{(1,3)} \\ 0_{(1,3)} & (\mathbf{r}_{\text{OB}_4} \times \hat{\mathbf{l}}_4)^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad (16)$$

respectively, where $\hat{\mathbf{l}}_i = \widehat{\mathbf{r}_{\mathbf{A}_i \mathbf{B}_i}}$ and both $\hat{\mathbf{l}}_i$ and \mathbf{r}_{OB_i} are functions of the mechanism pose \mathbf{q} with $i = 1 \dots 4$. Let the task of the manipulator be described in terms of the vector y in Figure 1 with forward kinematics of the form of (13) where

$$h(\mathbf{q}) = \cos(q_1) + \cos(q_1 + q_2). \quad (17)$$

Defining the manipulator state $\mathbf{x} \in \mathbb{R}^4$, to be given by $\mathbf{x} = [\mathbf{x}_1^T \ \mathbf{x}_2^T]^T = [\mathbf{q}^T \ \dot{\mathbf{q}}^T]^T$, the system input to be $\mathbf{u} = \mathbf{f} \in \mathbb{R}^4$ and the system output to be given by y , the dynamics (11) of the robot can be represented in the state space form

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_2 \\ -M(\mathbf{x}_1)^{-1} (\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2) + L^T(\mathbf{x}_1)\mathbf{u}) \end{bmatrix}, \quad (18)$$

where M , \mathbf{C} and L are given by equations (14), (15) and (16), respectively and $M(\mathbf{q})$ is always non-singular. The system (18) is subject to the unilateral constraint

$$\mathbf{u} \geq \mathbf{0}, \quad (19)$$

and has an output equation given by (13) with $h(\mathbf{x})$ given by (17).

B. Linearised Cable Robot Dynamics

The cable robot dynamics (11) are a continuous nonlinear time invariant system such that linearisation can be used to evaluate the positive controllability of the system through Proposition 4. Taking the linearisation about the equilibrium triple $(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}}, \mathbf{y}^{\text{eq}}) = ([\mathbf{q}^{\text{eq}}, \mathbf{0}]^T, \mathbf{u}^{\text{eq}}, \mathbf{y}^{\text{eq}})$ it can be seen that the nominal linearised model is given by

$$\begin{aligned} \delta \dot{\mathbf{x}} &= A \delta \mathbf{x} + B \delta \mathbf{u} \\ \delta \mathbf{y} &= C \delta \mathbf{x}, \end{aligned} \quad (20)$$

where the state, input and output matrices of the linearised system are given by

$$A = \begin{bmatrix} 0_{(2 \times 2)} & I_2 \\ \frac{\partial}{\partial \mathbf{x}_1} (-M(\mathbf{x}_1)^{-1} L^T(\mathbf{x}_1)\mathbf{u})|_{(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})} & 0_{(2 \times 2)} \end{bmatrix}, \quad (21)$$

$$B = \begin{bmatrix} 0_{(2 \times 4)} \\ -M(\mathbf{q}^{\text{eq}})^{-1} L^T(\mathbf{q}^{\text{eq}}) \end{bmatrix} \quad (22)$$

and

$$C = \begin{bmatrix} \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}}|_{(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}})} & 0_{(1 \times 2)} \end{bmatrix}, \quad (23)$$

respectively, where M , L and h are given by (14), (16) and (17).

Let the equilibrium triple represent the nominal operating point $(\mathbf{x}^{\text{eq}}, \mathbf{u}^{\text{eq}}, \mathbf{y}^{\text{eq}}) = ([\frac{\pi}{6} \ \frac{\pi}{3} \ 0 \ 0]^T, \mathbf{0}, \frac{\sqrt{3}}{2})$. Substitution of this nominal operating point into the expressions for the state matrix (21), input matrix (22) and output matrices (23) results in

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (24)$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.500 & -0.046 & -0.337 & -0.822 \\ -1.000 & 3.088 & 0.674 & 2.310 \end{bmatrix} \quad (25)$$

and

$$C = [-1.5 \ -1 \ 0 \ 0] \quad (26)$$

respectively.

C. Positive Controllability of the Nominal Operating Point

The positive output controllability of the linearisation (20) can be evaluated utilising Theorem 1 whereby from the given form for the state matrix (24) it can be seen that

$$e^{At} = \begin{bmatrix} I_2 & tI_2 \\ 0_{2 \times 2} & I_2 \end{bmatrix}, \quad (27)$$

such that

$$C e^{At} B \mathbf{u} = t [0.250 \ -3.018 \ -0.169 \ -1.077] \mathbf{u}. \quad (28)$$

As a result, the inner product $\langle v, C e^{At} B \mathbf{u} \rangle$ is given by

$$\langle v, C e^{At} B \mathbf{u} \rangle = -tv [-0.250 \ 3.018 \ 0.169 \ 1.077] \mathbf{u}. \quad (29)$$

If $v > 0$, then for all time $t > 0$ the choice of input $\mathbf{u} = [1 \ 0 \ 0 \ 0]^T$ results in the inner product (29) being positive. Similarly if $v < 0$, the choice of input $\mathbf{u} = [0 \ 1 \ 0 \ 0]^T$ results in (29) being positive for all time $t > 0$. This means that there is no non-zero v such that $\langle v, C e^{At} B \mathbf{u} \rangle$ is non-positive for all times $t > 0$ and positive input $\mathbf{u} \geq \mathbf{0}$. As a result the nominal operating point has a positive output controllable linearisation by Theorem 1.

Remark 11: Despite being positive output controllable, the linearised system is not positive controllable. This is

because the vector $\mathbf{v} = [0 \ 0 \ -2 \ -1]$ is an eigenvector of A^T , where A is given by (24), for which

$$\langle \mathbf{v}, B\mathbf{u} \rangle = -2.995u_2 - 0.666u_4. \quad (30)$$

For all $\mathbf{u} \geq \mathbf{0}$ the inner product (30) is negative such that the second equivalent condition of Proposition 1 is not satisfied and therefore the linearised system is not positive controllable. \circ

Utilising the same linearisation procedure for the equilibrium poses in the vicinity of the nominal operating point $([\frac{\pi}{6} \ \frac{\pi}{3} \ 0 \ 0]^T, \mathbf{0}, \frac{\sqrt{3}}{2})$ it can be seen that all other linearisations are positive output controllable. Since there is a ball of positive output controllable points surrounding the nominal operating point, then the point must connect to another positive output controllable linearisation such that $([\frac{\pi}{6} \ \frac{\pi}{3} \ 0 \ 0]^T, \mathbf{0}, \frac{\sqrt{3}}{2})$ belongs to a positive output controllable region.

Remark 12: For the chosen nominal operating point it can be seen that the positive output controllability of the system corresponded to the ability for the cable actuators to provide acceleration in every direction and magnitude of of the output space. This result is similar to the existing wrench closure condition [9], typically utilised in the trajectory generation and control of cable driven robots, which requires that force can be produced in every direction and magnitude of the output space. Noting that a non-zero equilibrium cable force input would result in a state matrix of the form of (21), it can be seen that it is possible for the nominal operating point to be positive output controllable without the mechanism being able to produce acceleration in every direction and magnitude of the output space at the nominal point. Evaluation of this case utilising Theorem 1 requires consideration of the matrix exponential e^{At} for the state matrix (21). This is a more complicated evaluation and has not been considered within this paper. \circ

Remark 13: In this example, the special form of the matrix A allows for Theorem 1 to be efficiently evaluated for all times $t > 0$ and $\mathbf{u} \geq \mathbf{0}$. For a general state matrix A , the solution for the matrix exponential e^{At} may not be easy to compute. This means that the evaluation of Theorem 1 could prove to be computationally expensive in a more general case. The derived condition is similar to the time dependent condition of Proposition 1. The potential for computational complexity motivates the derivation of time independent conditions analogous to the second equivalent condition from Proposition 1 as future work which will allow for more computationally efficient evaluation of positive output controllability. \circ

V. CONCLUSION

In this paper the positive output controllability of a multi-link cable driven parallel manipulator was considered. For simplicity, a condition on the positive output controllability of continuous LTI systems were derived. When applied to the multi-link cable driven parallel manipulator, the local positive

output controllability condition is then obtained through linearisation. The condition was verified on an example multi-link cable driven manipulator. Future work will look at determining equivalent time independent conditions for the positive output controllability of continuous LTI systems that are computationally more efficient.

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