Minimal Models of Spaces

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1 Conventions

Throughout this note graded objects are $\mathbb{Z}_{\geq 0}$-graded. We work with the category of commutative-graded $\mathbb{Q}$-algebras $\mathbf{CGA}_{\mathbb{Q}}$ and the category of differential commutative-graded $\mathbb{Q}$-algebras $\mathbf{CDGA}_{\mathbb{Q}}$. $\mathbb{Q}$ is viewed as an object of $\mathbf{CGA}_{\mathbb{Q}}$ concentrated in degree 0 and objects of $\mathbf{CDGA}_{\mathbb{Q}}$ are often specified as pairs $(\mathcal{A}^\bullet, d)$ where $\mathcal{A}^\bullet \in \mathbf{CGA}_{\mathbb{Q}}$ and $d : \mathcal{A}^\bullet \to \mathcal{A}^\bullet$ a map of $\mathbb{Q}$-vectors spaces satisfying $d(\mathcal{A}^n) \subseteq \text{Cocycles}(\mathcal{A}^{n+1})$ for all $n \geq 0$.

Write $\mathbf{GMod}_{\mathbb{Q}}$ for the category of graded $\mathbb{Q}$-vectors spaces. The forgetful functor $\mathbf{CGA}_{\mathbb{Q}} \to \mathbf{GMod}_{\mathbb{Q}}$ admits a left adjoint $V \mapsto \Lambda(V_\bullet) := S(V^{\text{even}}) \otimes E(V^{\text{odd}})$, where $S(V^{\text{even}})$ denotes the symmetric algebra on the even degree part of V and $E(V^{\text{odd}})$ denotes the exterior algebra on the odd degree part of V. In life, vector spaces tend to have bases, to keep track of these under $\Lambda(-)$ we write things like $\Lambda(x_i, y_j, z_k)$ which is the same as $\Lambda(V)$ with $V = V_i \oplus V_j \oplus V_k$, $V_i = \mathbb{Q}\text{-span}\{x\}$, $V_j = \mathbb{Q}\text{-span}\{y\}$, and $V_k = \mathbb{Q}\text{-span}\{z\}$.

A graded vector space $V_\bullet$ is of *finite type* if V_n is finite dimensional for all n.

2 Rational Homotopy Groups via Minimal Models

Recall that if $(\mathcal{M}^\bullet, d) \in \mathbf{CDGA}_{\mathbb{Q}}$ is a Sullivan algebra (in the sense of ([1] Definition 1.10)) then $\mathcal{M}^\bullet = \Lambda(V^\bullet)$ for some $V_\bullet \in \mathbf{GMod}_{\mathbb{Q}}$ with $V_0 = V_1 = 0$.

Lemma 2.1. *Let $\mathcal{A} \in \mathbf{CDGA}_{\mathbb{Q}}$ be such that $H^\bullet(\mathcal{A})$ is of finite type. Suppose $(\mathcal{M}, d) \to \mathcal{A}$ is a Sullivan minimal model (in the sense of ([1]. Definition 1.10)), then there exists $V_\bullet \in \mathbf{GMod}_{\mathbb{Q}}$ unique up to isomorphism such that $\mathcal{M}^\bullet \cong \Lambda(V_\bullet)$.*

Proof. Let $(\Lambda(V_\bullet), d) \to \mathcal{A}$ be the Sullivan minimal model of $\mathcal{A}$ constructed in ([2], Theorem. 10.3). Since $H^\bullet(\mathcal{A})$ is of finite type, properties of the construction in ([2], Theorem. 10.1) imply $V_\bullet$ is of finite type. Suppose $(\Lambda(V), d) \to \mathcal{A}$ is another Sullivan minimal model, then uniqueness of Sullivan minimal models ([1], Proposition. 1.18.) implies $\Lambda(V) \cong \Lambda(W)$. Since $\Lambda(V) \cong \Lambda(W)$ there exists $m_1 \in \mathbb{Z}_{\geq 0}$ minimal such that $\Lambda^{m_1}(V) \neq 0 \neq \Lambda^{m_1}(W)$. Since m_1 is minimal $\Lambda^{m_1}(V) = V_{m_1}$ and $\Lambda^{m_1}(W) = W_{m_1}$. Since $\Lambda(V) \cong \Lambda(W)$ preserves the grading, $V_{m_1} \cong W_{m_1}$. Now let $m_2 \in \mathbb{Z}_{>m_1}$ be minimal such that $\Lambda^{m_2}(V) \neq 0 \neq \Lambda^{m_2}(W)$. Since $V_\bullet$ is finite dimensional in each degree, one can count dimensions in the isomorphism $\Lambda^{m_1+m_2}(V_\bullet) \cong \Lambda^{m_1+m_2}(W_\bullet)$ to show that $V_{m_2} \cong W_{m_2}$. $\square$

Remark 2.2. Let K be a simply connected, simplicial complex such that $H^\bullet(K;\mathbb{Q})$ is of finite type. Fix a Sullivan minimal model $(\mathcal{M}^\bullet, d) \to A^\bullet(K)$ where $A^\bullet(K) \in \mathbf{CDGA}_\mathbb{Q}$ denotes the piece-wise linear deRham complex of K (in the sense of ([2], §9.1)). By ([2], Theorem 9.1) we know $H^\bullet(A^\bullet(K))$ is of finite type. So by Lemma (2.1), we can write $\mathcal{M}^\bullet = \Lambda(V_\bullet)$ where $V_\bullet$ is determined up to isomorphism by K. In otherwords, the graded vector space $V_\bullet$ is an invariant of K. The next result inteprets $V_\bullet$ in a familiar context.

Theorem 2.3. *Let K be a simply connected, simplicial complex such that $H^\bullet(K;\mathbb{Q})$ is of finite type. Fix a Sullivan minimal model $(\Lambda(V_\bullet), d) \to A^\bullet(K)$. Then there is an isomorphism in* $\mathbf{GMod}_\mathbb{Q}$*,*

$$\pi_\bullet(K) \otimes \mathbb{Q} \cong \mathrm{Hom}_\mathbb{Q}(V_\bullet, \mathbb{Q}).$$

In particular, if there exists a zig-zag of quasi-isomorphism in $\mathbf{CDGA}_\mathbb{Q}$ *of the form*

$$A^\bullet(K) \xleftarrow{\sim} \mathcal{A}^\bullet \xrightarrow{\sim} (H^\bullet(K;\mathbb{Q}), 0),$$

then the graded vector space $\pi_\bullet(K) \otimes \mathbb{Q}$ is formally determined by the graded algebra $H^\bullet(K;\mathbb{Q})$.

Remark 2.4. Simplicial complexes for which there is the zig-zag of quasi-isomorphism above are termed *formal* ([1]. Definition. 2.1). In this note we will see that S^n for $n \geq 2$ is formal. Formal spaces are significant since their rational cohomology abstractly determines their rational homotopy. For a more elementary example of a similar phenomena, let X be a space, G an abelian group, and consider the split exact given by the Universal Coefiicient Theorem

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X;G) \to \mathrm{Hom}(H_n(X), G) \to 0.$$

Since the sequence is split we obtain an isomorphism $H^n(X;G) \cong \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}(X), G)$, which says that as a graded G-module *the cohomology of X with arbitrary coefficients is formally determined by the integral homology.* Of course the integral homology does not determine the cup product structure on $H^\bullet(X;G)$. One can see this comparing the spaces $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$.

3 Examples: $\pi_i(S^n) \otimes \mathbb{Q}$

We demonstrate the utility of Theorem (2.3) by using it to compute the graded vector spaces $\pi_\bullet(K) \otimes \mathbb{Q}$ for $K = S^n$ with $n \geq 2$.

Example 3.1. $K = S^{2k+1}$ *for* $k > 1$*;*

By ([2], Thm. 9.1), $H^\bullet(A^\bullet(K)) \cong H^\bullet(S^{2k+1};\mathbb{Q})$ in $\mathbf{CGA}_\mathbb{Q}$*. Since $H^\bullet(S^{2k+1};\mathbb{Q}) \cong \mathbb{Q}[x]/(x^2)$ with $|x| = 2k+1$ and $\mathbb{Q}[x]/(x^2) \cong \Lambda(x_{2k+1})$, we obtain a zig-zag of quasi-isomorphisms in* $\mathbf{CDGA}_\mathbb{Q}$

$$A^\bullet(K) \xleftarrow{\sim} (\Lambda(x_{2k+1}), 0) \xrightarrow{\sim} (H^\bullet(S^{2k+1};\mathbb{Q}), 0)$$

where the right facing arrow is an isomorphism and the left facing arrow is given by sending x to a generator of cohomology in $A^{2k+1}(K)$. Hence S^{2k+1} is formal and by Theorem (2.3)

$$\pi_\bullet(S^{2k+1}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \bullet = 0, 2k+1, \\ 0 & \textit{else.} \end{cases}$$

In particular $\pi_i(S^{2k+1})$ is torsion unless $i = 0$ or $i = 2k+1$.

Example 3.2. *$K = S^{2k}$ for $k \geq 1$;*

Again by ([2], Theorem. 9.1), $H^{\bullet}(A^{\bullet}(K)) \cong H^{\bullet}(S^{2k};\mathbb{Q})$ in $\mathbf{CGA}_{\mathbb{Q}}$. Since $H^{\bullet}(S^{2k};\mathbb{Q}) \cong \mathbb{Q}[x]/(x^2)$ with $|x| = 2k$ we are tempted to write $H^{\bullet}(A^{\bullet}(K)) \cong (\Lambda(x_{2k}), 0)$, however this is not true since $H^{4k}(A^{\bullet}(K)) = 0$ whereas $x^2 \in \Lambda^{4k}(x_{2k})$ and so $\Lambda^{4k}(x_{2k}) \neq 0$. To remedy this consider $\Lambda(x_{2k}, y_{4k-1}) \in \mathbf{CGA}_{\mathbb{Q}}$. Since $S(x_{2k}) = \oplus_{k\in\mathbb{Z}_{\geq 0}}\mathbb{Q}\text{-span}\{x^k\}$ and $E(y_{4k-1}) = \mathbb{Q}\text{-span}\{y\}$ we get

$$\Lambda^m(x_{2k}, y_{4k-1}) = \bigoplus_{i+j=m} S^i(x_{2k}) \otimes E^j(y_{4k-1}) = \begin{cases} \mathbb{Q}\text{-span}\{x^{\ell} \otimes 1\} & \text{if } m = 2\ell k \text{ with } \ell \geq 1, \\ \mathbb{Q}\text{-span}\{x^{\ell-2} \otimes y\} & \text{if } m = 2\ell k - 1 \text{ with } \ell \geq 2, \\ 0 & \text{else.} \end{cases}$$

So,

$$d := \begin{cases} d(x^{\ell-2} \otimes y) = x^{\ell} \otimes 1 & \text{if } m = 2\ell k - 1 \text{ with } \ell \geq 2 \\ 0 & \text{else,} \end{cases}$$

defines a differential $d : \Lambda^m(x_{2k}, y_{4k-1}) \to \Lambda^{m+1}(x_{2k}, y_{4k-1})$ and we obtain a zig-zag of quasi-isomorphisms

$$A^{\bullet}(K) \xleftarrow{\sim} (\Lambda(x_{2k}, y_{4k-1}), d) \xrightarrow{\sim} (H^{\bullet}(S^{2k};\mathbb{Q}), 0).$$

The left facing arrow maps x to a fixed generator of cohomology $\omega \in A^{2k}(K)$ and maps y to any element $\eta \in A^{4k-1}(K)$ such that $d\eta = \omega \wedge \omega$. Such an η exists since $\omega \wedge \omega$ is closed and therefore exact since $H^{4k}(A^{\bullet}(K)) = 0$. The right facing arrow maps x to the unique generator of $(H^{\bullet}(S^{2k};\mathbb{Q})$ and maps y to zero. Hence S^{2k} is formal and Theorem (2.3) implies

$$\pi_{\bullet}(S^{2k}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \bullet = 0, 2k, 4k-1 \\ 0 & \text{else.} \end{cases}$$

In particular $\pi_i(S^{2k})$ is torsion unless $i = 0, 2k, 4k-1$.

4 Reduction of Theorem (2.3)

The statement of Theorem (2.3) is presented with a view to computing $\pi_{\bullet}(K) \otimes \mathbb{Q}$ for real life spaces. However the proof of of Theorem (2.3) proceeds by a reductions to Proposition (4.1). Before giving the statement of Proposition (4.1) let us first recall a definition.

Definition. Let $\mathcal{A} \in \mathbf{CDGA}_{\mathbb{Q}}$, $n \geq 1$, and $V_{\bullet} \in \mathbf{GMod}_{\mathbb{Q}}$ concentrated in degree n. A *n-Hirsch extension of $\mathcal{A}$ by V* is a diagram in $\mathbf{CDGA}_{\mathbb{Q}}$ of the form $\mathcal{A} \hookrightarrow \mathcal{H}$ together with a differential $d : \mathcal{A} \otimes \Lambda(V) \to \mathcal{A} \otimes \Lambda(V)$ such that $\mathcal{H} \cong (\mathcal{A} \otimes \Lambda(V), d)$ in $\mathbf{CDGA}_{\mathbb{Q}}$.

Two n-Hirsch extension $\mathcal{A} \to (\mathcal{A} \otimes \Lambda(V), d)$ and $\mathcal{A} \to (\mathcal{A} \otimes \Lambda(W), \delta)$ are *equivalent*, written $(\mathcal{A} \otimes \Lambda(V), d) \sim (\mathcal{A} \otimes \Lambda(W), \delta)$, if there is a commutative diagram in $\mathbf{CDGA}_{\mathbb{Q}}$

$$\begin{tikzcd} & \mathcal{A} & \\ (\mathcal{A} \otimes \Lambda(V), d) \arrow[ur] \arrow[rr, "\cong"] & & (\mathcal{A} \otimes \Lambda(W), \delta). \arrow[ul] \end{tikzcd}$$

Proposition 4.1. *Let X be a simply connected, simplicial complex such that $H^\bullet(X;\mathbb{Q})$ is of finite type. $H^\bullet(X;\mathbb{Q})$. Assume $\pi_n(X)$ is a finite dimensional $\mathbb{Q}$-vector space for all n. Then there exists an inductive system*

$$\begin{tikzcd}
 & & A^\bullet(K) & & \\
\mathcal{M}(2) \arrow[r, hook] \arrow[urr, bend left, "\mu_2"] & \mathcal{M}(3) \arrow[r, hook] \arrow[ur, "\mu_3"] & \dots \arrow[r, hook] & \mathcal{M}(n) \arrow[r, hook] \arrow[ul, "\mu_n"'] & \dots
\end{tikzcd}$$

such that:

- *(a)* $\mathcal{M}(2) = (\Lambda(\pi_2(X)^*), 0)$.
- *(b)* $\mathcal{M}(n-1) \hookrightarrow \mathcal{M}(n)$ *is an n-Hirsch extension with* $\mathcal{M}(n) = (\mathcal{M}(n-1) \otimes \Lambda(\pi_n(X)^*), d_n)$.
- *(c)* μ_n *is n-minimal i.e.* $\mu_n^* : H^i(\mathcal{M}(n)) \to H^i(A^\bullet(X))$ *is an isomorphism for* $i \leq n$.

Proof that (3.1) implies (2.1):

Assume the conclusion of Proposition (4.1) and suppose K satisfies the hypothesis of Theorem (2.3). Let $\ell : K \to K_0$ denote the localization of K at 0 and let X be a simplicial complex homotopy equivalent to K_0. Since $H^\bullet(K;\mathbb{Q})$ is of finite type, ([1], Definition 1.7) implies $H^\bullet(X;\mathbb{Q})$ is of finite type. So by ([1], Definition. 1.1) the graded $\mathbb{Q}$-vector space $\pi_\bullet(X)$ is of finite type [???]. So X satisfies the hypothesis of Proposition (4.1) and we obtain an inductive system

$$\begin{tikzcd}
 & & A^\bullet(X) & & \\
\mathcal{M}(2) \arrow[r, hook] \arrow[urr, bend left, "\mu_2"] & \mathcal{M}(3) \arrow[r, hook] \arrow[ur, "\mu_3"] & \dots \arrow[r, hook] & \mathcal{M}(n) \arrow[r, hook] \arrow[ul, "\mu_n"'] & \dots
\end{tikzcd}$$

By the proof of ([2], Theorem. 10.3) there exists a differential $d : \Lambda(\pi_\bullet(X)^*) \to \Lambda(\pi_\bullet(X)^*)$ such that $\varinjlim(\mathcal{M}(n)) \cong (\Lambda(\pi_\bullet(X)^*), d)$. The composition $\pi_\bullet(K) \otimes \mathbb{Q} \to \pi_\bullet(K_0) \otimes \mathbb{Q} \cong \pi_\bullet(X)$ is an isomorphism so $\varinjlim(\mathcal{M}(n)) = (\Lambda((\pi_\bullet(K) \otimes \mathbb{Q})^*), d)$. Hence by Remark (2.2) all that remains is to show $\varinjlim(\mathcal{M}(n))$ is a Sullivan minimal model for $A^\bullet(K)$. By the proof of ([2], Theorem. 10.3) it will suffice to construct maps $\tilde{\mu}_n : \mathcal{M}(n) \to A^\bullet(K)$ such that the diagram commutes

$$\begin{tikzcd}
 & & A^\bullet(K) & & \\
\mathcal{M}(2) \arrow[r, hook] \arrow[urr, bend left, "\tilde{\mu}_2"] & \mathcal{M}(3) \arrow[r, hook] \arrow[ur, "\tilde{\mu}_3"] & \dots \arrow[r, hook] & \mathcal{M}(n) \arrow[r, hook] \arrow[ul, "\tilde{\mu}_n"'] & \dots
\end{tikzcd}$$

and $\tilde{\mu}_n$ is n-minimal i.e. $\tilde{\mu}_n^* : H^i(\mathcal{M}(n)) \to H^i(A^\bullet(K))$ is an isomorphism for $i \leq n$.

Let $\ell_s : K' \to X$ denote a simplicial approximation to the composition $K \xrightarrow{\ell} K_0 \simeq X$. Since $H^\bullet(\ell_s;\mathbb{Q})$ is an isomorphism, ([2], Theorem. 9.1) implies that the induced map $A^\bullet(X) \to A^\bullet(K')$ is a quasi-isomorphism. Since K' is a subdivision of K, $K' \xrightarrow{\text{id}} K$ is a simplicial map. By ([2], Theorem. 9.1) the induced map $A^\bullet(K) \xrightarrow{\text{id}^*} A^\bullet(K')$ is a quasi-isomorphism which implies that $H^\bullet(A^\bullet(K), A^\bullet(K'))$ vanishes. Let $\tilde{\mu}_2$ be the lift in ([2], Proposition. 11.1) for the commuting square

$$\begin{tikzcd}
(\mathbb{Q},0) \arrow[r] \arrow[d] & A^{\bullet}(K) \arrow[d] \\
\mathcal{M}(2) \arrow[r] \arrow[ur, dashed, "\tilde{\mu}_2"] & A^{\bullet}(K')
\end{tikzcd}$$

So $\mathcal{M}(2) \xrightarrow{\tilde{\mu}_2} A^{\bullet}(K) \to A^{\bullet}(K')$ is homotopic to $\mathcal{M}(2) \to A^{\bullet}(X) \to A^{\bullet}(K')$. Since μ_2 is 2-minimal and $A^{\bullet}(X) \to A^{\bullet}(K')$ is a quasi-isomorphism, $\tilde{\mu}_2$ is 2-minimal. The diagram commutes up to homotopy.

$$\begin{tikzcd}
\mathcal{M}(2) \arrow[rr, "\tilde{\mu}_2"] \arrow[d] \arrow[dr] & & A^{\bullet}(K) \arrow[d] \\
\mathcal{M}(3) \arrow[r, "\mu_3"] & A^{\bullet}(X) \arrow[r] & A^{\bullet}(K')
\end{tikzcd}$$

So ([2], Proposition. 11.1) implies that $\tilde{\mu}_2$ extends to a map $\tilde{\mu}_3 : \mathcal{M}(3) \to A^{\bullet}(K)$ such that $\mathcal{M}(3) \xrightarrow{\tilde{\mu}_3} A^{\bullet}(K) \to A^{\bullet}(K')$ is homotopic to $\mathcal{M}(3) \xrightarrow{\mu_3} A^{\bullet}(X) \to A^{\bullet}(K')$. Since μ_3 is 3-minimal and $A^{\bullet}(X) \to A^{\bullet}(K')$ is a quasi-isomorphism, $\tilde{\mu}_3$ is 3-minimal. Continuing in this way we obtain the family $\{\tilde{\mu}_n\}_{n\geq 2}$ as required. $\square$

5 The tower $\mathcal{M}(2) \hookrightarrow \mathcal{M}(3) \hookrightarrow \ldots$

In this section we embark on the proof of Proposition (4.1), to be precise we construct the tower $\mathcal{M}(2) \hookrightarrow \mathcal{M}(3) \hookrightarrow \ldots$. The main technical input to the construction is a duality result connecting minimal models of $A^{\bullet}(X)$ to the Postnikov tower of X. Proposition (5.1) and Theorem (5.2) give precise statements of this duality.

Proposition 5.1. *([2], §12.1) Let B be a simplicial complex, $n \geq 1$, and π an graded abelian group concentrated in degree n such that $V := \pi \otimes \mathbb{Q}$ is a finite dimensional $\mathbb{Q}$ vector space. Fix a Sullivan minimal model $\mathcal{M} \to A^{\bullet}(B)$. Then there is a bijection*

$$\cong \Big\backslash \left\{ \begin{matrix} k(\pi,n)\text{-fibrations} \\ E \to B \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} n\text{-Hirsch extensions} \\ \mathcal{M} \hookrightarrow (\mathcal{M} \otimes \Lambda(V^*), d) \end{matrix} \right\} \Big/ \sim$$

where $\cong$ on the left denotes fiber-wise homotopy equivalence.

Suppose $\mathcal{M} \hookrightarrow (\mathcal{M} \otimes \Lambda(V^*), d_f)$ is the Hirsch extension corresponding to a $k(\pi,n)$-fibration $f : E \to B$. One can ask whether the diffential graded algebra $(\mathcal{M} \otimes \Lambda(V^*), d_f)$ is related more directly to the topology of the fibration $f : E \to B$. The following result ([2], Theorem. 12.1.) answers this question in the affirmative. Its proof forms the technical heart of Theorem (2.3).

Theorem 5.2. *$(\mathcal{M} \otimes \Lambda(V^*), d_f)$ is naturally a minimal Sullivan model for $A^{\bullet}(E')$. More precisely, if $f_s : E' \to B$ is a simplicial approximation for f. Then there is a Sullivan minimal model $\rho : (\mathcal{M} \otimes \Lambda(V^*), d_f) \to A^{\bullet}(E')$ commuting the diagram*

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d] & A^{\bullet}(B) \arrow[d, "f_s^*"] \\
(\mathcal{M} \otimes \Lambda(V^*), d_f) \arrow[r, "\rho"] & A^{\bullet}(E').
\end{tikzcd}$$

Let $\ldots \xrightarrow{p_5} X_4 \xrightarrow{p_4} X_3 \xrightarrow{p_3} X_2$ be a simplicial model for the Postnikov tower of X. So each p_i is as simplicial approximation to a $k(\pi_i(X), i)$-fibration and $X_2 \simeq k(\pi_2(X), 2)$. In the appendix we construct a Sullivan minimal model $\Lambda(\pi_2(X)^*), 0) \to A^\bullet(X_2)$. Appyling Theorem (5.2) with $\mathcal{M} = (\pi_2(X)^*), 0)$ and f the $k(\pi_3(X), 3)$-fibration corresponding to p_3 gives a commutative diagram

$$\begin{tikzcd}
\Lambda(\pi_2(X)^*), 0) \arrow[r] \arrow[d] & A^\bullet(X_2) \arrow[d, "p_3^*"] \\
(\Lambda(\pi_2(X)^*) \otimes \Lambda(\pi_3(X)^*), d) \arrow[r, "\rho_2"] & A^\bullet(X_3)
\end{tikzcd}$$

in which ρ_3 is a Sullivan minimal model. Now set $\mathcal{M}(2) = \Lambda(\pi_2(X)^*), 0)$, $\mathcal{M}(3) = (\Lambda(\pi_2(X)^*) \otimes \Lambda(\pi_3(X)^*), d)$. Iteratively applying Theorem (5.2) in this way yields a diagram

$$\begin{tikzcd}
A^\bullet(X_2) \arrow[r, "p_3^*"] & A^\bullet(X_3) \arrow[r, "p_4^*"] & \ldots \arrow[r, "p_n^*"] & A^\bullet(X_n) \arrow[r, "p_{n+1}^*"] & \ldots \\
\mathcal{M}(2) \arrow[r, hook] \arrow[u, "\rho_2"] & \mathcal{M}(3) \arrow[r, hook] \arrow[u, "\rho_3"] & \ldots \arrow[r, hook] & \mathcal{M}(n) \arrow[r, hook] \arrow[u, "\rho_n"] & \ldots
\end{tikzcd}$$

and by contruction the bottom row satisfies conditions (a) and (b) of Proposition (4.1).

6 The maps μ_n.

In this section we conclude the proof of Proposition (4.1) by constructing the μ_n's. Keeping notation as in section 5 let $\pi_2 : X(2) \to X_2$ be a simplicial approximation to the canonical projection $X \to X_2$. Since $X(2)$ is a subdivision of X the identity, $X(2) \to X$ is a simplicial map and we obtain a commutative square

$$\begin{tikzcd}
(\mathbb{Q}, 0) \arrow[r] \arrow[d] & A^\bullet(X) \arrow[d] \\
\mathcal{M}(2) \arrow[r] & A^\bullet(X(2))
\end{tikzcd}$$

where the bottom arrow is the composition $\mathcal{M}(2) \xrightarrow{\rho_2} A^\bullet(X_2) \xrightarrow{\pi_2^*} A^\bullet(X(2))$. Since $X(2) \to X$ is the identity map, ([2], Theorem 9.1) implies $A^\bullet(X) \to A^\bullet(X(2))$ is a quasi-isomorphism. So the relative cohomology $H^\bullet(A^\bullet(X), A^\bullet(X(2)))$ vanishes. Therefore we can apply ([2], Propostion. 11.1) to the square above and obtain $\mu_2 : \mathcal{M}(2) \to A^\bullet(X)$ such that the composition $\mathcal{M}(2) \xrightarrow{\mu_2} A^\bullet(X) \to A^\bullet(X(2)))$ is homotopic to the composition $\mathcal{M}(2) \xrightarrow{\rho_2} A^\bullet(X_2) \xrightarrow{\pi_2^*} A^\bullet(X(2))$.

Since π_2 is a simplicial model for the the canonical projection $X \to X_2$, we know π_2^* is 2-minimal. Since ρ is a quasi-ismorphism it follows that $\mathcal{M}(2) \xrightarrow{\rho_2} A^\bullet(X_2) \xrightarrow{\pi_2^*} A^\bullet(X(2))$ is 2-minimal. Therefore μ_2 is 2-minimal since $\mathcal{M}(2) \xrightarrow{\mu_2} A^\bullet(X) \to A^\bullet(X(2)))$ is homotopic to $\mathcal{M}(2) \xrightarrow{\rho_2} A^\bullet(X_2) \xrightarrow{\pi_2^*} A^\bullet(X(2))$ and $A^\bullet(X) \to A^\bullet(X(2))$ is a quasi-isomorphism.

Next let $\pi_3 : X(3) \to X_3$ be a simplicial approximation to the canonical projection $X(2) \to X_3$. Since $X(3)$ is a subdivision of $X(2)$ the identity $X(3) \to X(2)$ is a simplicial map and we obtain a diagram which commutes up to homotopy:

$$\begin{tikzcd}
& A^\bullet(X) \arrow[dl] \arrow[dr] & \\
A^\bullet(X(3)) & & A^\bullet(X(2)) \arrow[ll] \\
A^\bullet(X_3) \arrow[u, "\pi_3^*"] & & A^\bullet(X_2) \arrow[ll] \arrow[u, "\pi_2^*"] \\
\mathcal{M}(3) \arrow[u] & & \mathcal{M}(2) \arrow[ll] \arrow[u] \arrow[uuul, bend right, "\mu_2"']
\end{tikzcd}$$

Similar to the construction μ_2, we can apply ([2], Proposition 11.1) to the commutative (up to homotopy) square

$$\begin{tikzcd}
\mathcal{M}(2) \arrow[r, "\mu_2"] \arrow[d] & A^\bullet(X) \arrow[d] \\
\mathcal{M}(3) \arrow[r] & A^\bullet(X(3))
\end{tikzcd}$$

and obtain a lift $\mathcal{M}(3) \to \mu_3 A^\bullet(X)$ such that the composition $\mathcal{M}(3) \xrightarrow{\mu_3} A^\bullet(X) \to A^\bullet(X(3)))$ is homotopic to the composition $\mathcal{M}(3) \xrightarrow{\rho_2} A^\bullet(X_3) \xrightarrow{\pi_2^*} A^\bullet(X(3))$. μ_3 is 3-minimal for the same reason μ_2 is 2-minimal.

7 Appendix

Keep notation as in §5. The purpose of this appendix is to construct a Sulivan minimal model of the form $(\Lambda(\pi_2(X)^*, 0)) \to A^\cdot(X_2)$. Consider the diagram in $\mathbf{CDGA}_\mathbb{Q}$

$$A^\cdot(X_2) \supseteq \text{Cocycles}(A^\cdot(X_2)) \twoheadrightarrow (H^\bullet(A^\bullet(X_2)), 0) \cong (H^\bullet(X_2; \mathbb{Q}), 0)$$

where the last isomorphism comes from ([2], Theorem. 9.1). Our strategy will be to identify $(H^\bullet(X_2; \mathbb{Q}), 0)$ with $(\Lambda(\pi_2(X)^*, 0))$, and then to lift the generators of $(\Lambda(\pi_2(X)^*, 0))$ through the composition above to obtain a quasi isomorphism $(\Lambda(\pi_2(X)^*, 0)) \to A^\cdot(X_2)$. By assumption $X_2 \simeq k(\pi_2(X), 2)$ and $\pi_2(X) \cong \mathbb{Q}^n$ for some $n \geq 0$. Hence

$$H^\bullet(X_2; \mathbb{Q}) \cong H^\cdot(k(\pi_2(X), 2); \mathbb{Q}) \cong H^\cdot(k(\mathbb{Q}^n, 2); \mathbb{Q}).$$

It is tautologically the case that $k(\mathbb{Q}^n, 2) \simeq k(\mathbb{Z}^n, 2)_0$. Also $(-)_0$ induces isomorphisms on rational cohomology. Therefore

$$H^\cdot(k(\mathbb{Q}^n, 2); \mathbb{Q}) \cong H^\cdot(k(\mathbb{Z}^n, 2)_0; \mathbb{Q}) \cong H^\cdot(k(\mathbb{Z}^n, 2); \mathbb{Q}).$$

Since $k(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$ and $\mathbb{C}P^\infty$ is path connected. It follows that $\pi_i(k(\mathbb{Z}, 2)^{\times n}) = \pi_i(k(\mathbb{Z}, 2))^{\times n} = \mathbb{Z}^n$ if $i = 2$ and 0 if $i \neq 2$. Therefore $k(\mathbb{Z}^n, 2) \simeq k(\mathbb{Z}, 2)^{\times n}$. So

$$H^\cdot(k(\mathbb{Z}^n, 2); \mathbb{Q}) \cong H^\cdot(k(\mathbb{Z}, 2)^{\times n}; \mathbb{Q}) \cong H^\cdot(k(\mathbb{Z}, 2); \mathbb{Q})^{\otimes n},$$

where the last isomorphism follows from Kunneth formula. Since $k(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$ and $H^\bullet(\mathbb{C}P^\infty; \mathbb{Q}) = \mathbb{Q}[x]$ with $|x| = 2$, we obtain

$$H^\cdot(k(\mathbb{Z}, 2); \mathbb{Q})^{\otimes n} \cong \mathbb{Q}[x]^{\otimes n} \cong \mathbb{Q}[x_1, \ldots, x_n] \quad \text{with } |x_i| = 2.$$

By assumption $\pi_2(X) \cong \mathbb{Q}^n$ and so $H^\cdot(k(\mathbb{Z}, 2); \mathbb{Q})^{\otimes n} \cong \Lambda(\pi_2(X))$.

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