# MINIMAL MODELS FOR DIFFERENTIAL GRADED ALGEBRAS

ALEXANDER P. CLARK

## 1. Recap

Today I'll be talking about the notion of a minimal model for a DGA, following [GM13]. First though, let me give a quick recap of the main objects that we saw two weeks ago.

First we have the notion of a differential graded algebra. We'll typically only worry about working over $k = \mathbb{Q}$, given that this semester we're learning about rational homotopy theory, but one could let $k$ be a ring, and define a DGA as a $k$-module with the same conditions as below, but let us proceed:

**Definition 1.1.** A differential graded(-commutative) $k$-algebra $(\mathcal{A}, d)$ is a pair, consisting of a graded $\mathbb{Q}$-vector space $\mathcal{A} = \bigoplus_{p \in \mathbb{Z}_{\geqslant 0}} \mathcal{A}^p$ and a map $d : \mathcal{A}^* \to \mathcal{A}^{*+1}$ such that

(1) $d^2 = 0$ (i.e. $(\mathcal{A}, d)$ is a cochain complex - where we often treat $\mathcal{A}$ itself as this cochain complex),
(2) There is a graded-commutative multiplication $\mathcal{A}^p \otimes \mathcal{A}^q \to \mathcal{A}^{p+q}$ such that $\alpha\beta = (-1)^{pq}\beta\alpha$,
(3) The differential satisfies a graded-leibniz rule $d(\alpha\beta) = d(\alpha)\beta + (-1)^p \alpha d(\beta)$.

**Remark 1.2.** One often abuses notation, and simply refers to the DGA $\mathcal{A}$, when they mean $(\mathcal{A}, d)$. I'll feel free to make this abuse, unless confusion may arise.

Recall the example from last week $\mathcal{A} = (\mathbb{C}[x, y]/(x^2), d)$ with $x$ of degree 1, $y$ of degree 2, and where $d$ was defined by $d(x) = y$ and $d(y) = xy$. Let us denote the $k$-span of $\{x_1, \ldots, x_n\}$ by $k\langle x_1, \ldots, x_n\rangle$. We can write down the underlying cochain complex as follows:

$$\mathbb{C} \longrightarrow \mathbb{C}\langle x\rangle \longrightarrow \mathbb{C}\langle y\rangle \longrightarrow \mathbb{C}\langle xy\rangle \longrightarrow \mathbb{C}\langle y^2\rangle \longrightarrow \ldots$$

and then one can simply compute the cohomology of this complex, and denote this cohomology algebra by $H^*(\mathcal{A})$, which is a DGA with $d = 0$ (where, again, I won't bother to denote the differential in the notation for the cohomology algebra). Note that this example DGA has cohomology that is zero in every degree.

Next, let us recall the notion of a minimal DGA. Denote by $A^+ = \bigoplus_{p \in \mathbb{Z}_{>0}} A^p$, the positively graded part of $\mathcal{A}$.

**Definition 1.3.** The DGA $\mathcal{A}$ is minimal if 1) $\mathcal{A}$ is free as a graded-commutative algebra on generators of degree $\geqslant 2$, 2) $d(\mathcal{A}) \subset A^+ \wedge A^+$.

The first part of the definition says that $\mathcal{A}$ is the tensor product of polynomial algebras on generators of even degree, and exterior algebras on generators of odd degree. In fact, we can fix a notation to make this more convenient. Let $\Lambda_k(x_1, \ldots, x_n)$ denote the graded polynomial algebra over $\mathbb{Q}$ on the $x_1, \ldots, x_n$ if $k$ is even, and be the exterior algebra on the $x_1, \ldots, x_n$ if $k$ is odd. Then the first part simply says that $\mathcal{A}$ is a tensor product of some $\Lambda_{k_i}(x_1^i, \ldots, x_{m_i}^i)$'s. The example above we have $\Lambda_1(x) \otimes \Lambda_2(y) = \mathbb{C}[x, y]/(x^2)$, but this is not minimal, since $d(x) = y$, and $y$ is not the product of two generators with degrees $\geqslant 1$.

As an example of a DGA that *is* minimal, consider $\Lambda_1(x) \otimes_k \Lambda_2(y, z)$ where $d(x) = 0, d(z) = 0$, and $d(y) = xy$. We see that $d(y) = xy$, which is a product of generators in degrees 1 and 2.

Next we recall the notion of a Hirsch extension.

*Date*: 28th of March, 2019.

**Definition 1.4.** A Hirsch extension of a DGA $(\mathcal{A}, d)$ is a triple $(\mathcal{A} \otimes \Lambda_k(x_1, \ldots, x_n), \varphi, d')$ where $\varphi : \mathcal{A} \to \mathcal{A} \otimes \Lambda_k(x_1, \ldots, x_n)$ is a morphism of DGAs (a map of cochain complexes, which is also a map of commutative-graded algebras) where $d'$ is defined as $d'|_{\mathcal{A}} = d$, and we define $d'|_{\Lambda_k(x_1,\ldots,x_n)}$ so that it respects the graded-leibniz rule.

**Remark 1.5.** Two weeks ago we saw $\mathcal{A} \to \mathcal{A} \otimes \Lambda(V^k)$, where in our notation $\Lambda(V^k)$ is just $\Lambda_k(\text{basis}(V))$, so this is the same thing.

## 2. Minimal Models

Now, we can define the minimal model at last. The minimal model is a cofibrant replacement in the appropriate model category (which we may see later). The idea is completely in analogy (even in a mathematically precise sense) to the notion of a postnikov tower.

**Definition 2.1.** A minimal model for $A^*$ is a minimal DGA $\mathfrak{M}(A^*)$ such that there exists a map $\rho : \mathfrak{M}(A^*) \to A^*$ of DGAs inducing an isomorphism on cohomology.

**Remark 2.2.** This says that $\rho$ is a weak equivalence from the cofibrant replacement in the appropriate model category. i.e. these objects are isomorphism in the appropriate homotopy category!)

We will soon give a construction for the minimal model of any simply connected DGA.

Now, as motivation for the term *simply connected DGA*, recall that a topological space $X$ is said to be simply connected if $\pi_0(X) = \pi_1(X) = 0$. One can then show that $H^0(X, k) = k$ and $H^1(X, k) = 0$. With this, it is reasonable to (and we will) say that a $k$-DGA $\mathcal{A}$ is simply connected if $H^0(\mathcal{A}) = k$ and $H^1(\mathcal{A}) = 0$.

The last thing we need before starting the construction of the minimal model for a DGA, is the notion of relative cohomology.

If $C^* \to D^*$ is a morphism of cochain complexes. Then the mapping cone (homotopy cokernel) of $f$ is the cochain complex $M_f^*$ such that:

$$M_f^n = C^n \oplus D^{n-1},$$

and where $\delta : M_f^n \to M_f^{n+1}$ is given by:

$$\begin{bmatrix} \partial_C & 0 \\ f & -\partial_D \end{bmatrix}.$$

One can easily show that $\delta^2 = 0$. (If you haven't seen this notation for the differential before, you are simply 'acting' this matrix on column vectors $(c, d)$ for $(c, d) \in C^n \oplus D^{n-1}$.)

**Definition 2.3.** The relative cohomology $H^*(C, D)$ is defined to be $H^*(M_f^*)$.

The maps $(-i_2) : D^{*-1} \to M_f^*$ and $\pi_1 : M_f^* \to C^*$ commute with the coboundaries, and we obtain a long exact sequence on cohomology:

$$\cdots \longrightarrow H^n(C) \xrightarrow{f^*} H^n(D) \xrightarrow{\delta} H^{n+1}(C, D) \xrightarrow{i^*} H^{n+1}(C) \xrightarrow{f^*} \cdots$$

where for commutative squares, we get the obvious map between long exact sequences.

Finally, we can deal with the construction.

**Theorem 2.4.** If $\mathcal{A}$ is a simply connected DGA, then there exists a minimal model $\mathfrak{M}(\mathcal{A})$ for $\mathcal{A}$.

*Proof.* Let $\mathcal{A}$ be a simply connected DGA. To build the minimal model, we will first build a tower of Hirsch extensions

$$k = \mathfrak{M}(0) = \mathfrak{M}(1) \subset \mathfrak{M}(2) \subset \cdots$$

and *provide* maps $\rho_n : \mathfrak{M}(n) \to \mathcal{A}$ such that $\rho_n|_{\mathfrak{M}(k)} = \rho_k$ for $k \leqslant n$, and such that $\rho_n : \mathfrak{M}(n) \to \mathcal{A}$ is an $n$-minimal model, in that:

(1) $\mathfrak{M}(n)$ is minimal and generated by elements in degrees $\leqslant n$.

(2) $\rho_n^*$ is an isomorphism on cohomology in degrees $\leqslant n$.
(3) $\rho_n^*$ is an injection on cohomology in degree $n+1$.

For the base step of induction, note that $k$ is a DGA with $d = 0$, then we can take $\rho_1 : k \hookrightarrow \mathcal{A}$ as the map sending 1 to 1. This gives an isomorphism on cohomology in degree $\leqslant 1$ and an injection for $n = 2$ as required. Next, we induct; suppose that for some $n \geqslant 1$ we have constructed $\mathfrak{M}(n)$ and *provided* $\rho_n : \mathfrak{M}(n) \to \mathcal{A}$ satisfying the three conditions above. Then, the relative cohomology $H^i(\mathfrak{M}(n), \mathcal{A})$ vanishes for $i \leqslant n+1$. To see this, note that by assumption $\rho_n^*$ is an isomorphism on cohomology in degrees $\leqslant n$, and hence the relative cohomology $H^i(\mathfrak{M}(n), \mathcal{A}) = 0$ for $i \leqslant n$ by assumption. For the $i = n+1$ case, consider the long-exact sequence for relative cohomology (making use of the injectivity of $\rho_{n+1}^*$ and the isomorphism for $\rho_n^*$):

$$\cdots \longrightarrow 0 \longrightarrow H^n(\mathfrak{M}(n)) \xrightarrow{\sim} H^n(\mathcal{A}) \xrightarrow{0} H^{n+1}(\mathfrak{M}(n), \mathcal{A}) \xrightarrow{0} H^{n+1}(\mathfrak{M}(n)) \hookrightarrow \cdots$$

Construct $\mathfrak{M}(n+1)$: Let $V = H^{n+2}(\mathfrak{M}(n), \mathcal{A})$ and take $\mathfrak{M}(n+1) = \mathfrak{M}(n) \otimes \Lambda_{n+1}(\text{basis}(V))$.

Note that we are only altering the degree $n+1$ part of $\mathfrak{M}(n)$. I.e. if we treat $\mathfrak{M}(n)$ and $\mathfrak{M}(n+1)$ as cochain complexes, they agree in each degree, except degree $n+1$. This means that the cohomology of each lower degree is unchanged. But for this to be a DGA, I have to tell you what the differential is on $\mathfrak{M}(n+1)$. To define this, we need to define it on $V$ subject to the condition that for each $v \in V$, $dv \in \mathfrak{M}(n)^{n+2}$ is closed. Additionally we need to provide the map $\rho_{n+1} : \mathfrak{M}(n+1) \to \mathcal{A}$ extending $\rho_n$, i.e. we need to define $\rho_{n+1}|V$ subject to the condition that $\rho_n(dv) = d\rho_{n+1}(v)$ for each $v \in V$.

Since we are taking $k$ as a field (usually $\mathbb{Q}$), all our vector-spaces are free, and hence we can take a linear (in $v \in V$) splitting $s$:

$$0 \longrightarrow B^{n+2}(\mathfrak{M}(n), \mathcal{A}) \longrightarrow Z^{n+2}(\mathfrak{M}(n), \mathcal{A}) \underset{s}{\rightleftarrows} H^{n+2}(\mathfrak{M}(n), \mathcal{A}) \longrightarrow 0$$

□

This is equivalent to choosing $k$-linearly for $v \in H^{n+2}(\mathfrak{M}(n), \mathcal{A})$ cocycle representatives $(m_v, a_v) \in \mathfrak{M}(n)^{n+2} \oplus \mathcal{A}^{n+1}$. Looking at the mapping cone, we have for $\rho_n$:

$$\begin{bmatrix} d_{\mathfrak{M}(n)} & 0 \\ \rho_n & -d_{\mathcal{A}} \end{bmatrix},$$

and to be in the kernel, we must have $d_{\mathfrak{M}(n)}(m_v) = 0$ and $\rho_n(m_v) = d_{\mathcal{A}}(a_v)$.

Then let us define $d(v) = m_v$ and $\rho_{n+1}(v) = a_v$. In which case,

$$d^2(v) = d(m_v) = d_{\mathfrak{M}(n)}(v) = 0,$$
$$\rho_n(dv) = \rho_n(m_v) = d_{\mathcal{A}} a_v = d_{\mathcal{A}}(\rho_{n+1}(v))$$

and thus $\mathcal{M}(n+1)$ is a DGA , and the $\rho_{n+1}$ is a map of DGAs as required.

Lastly we need to show that $H^i(\mathfrak{M}(n+1), \mathcal{A}) = 0$ for $i \leqslant n+1$, and that we have an injection for $i = n+2$. To do this, we can show that 1) $H^{n+2}(\mathfrak{M}(n), \mathfrak{M}(n+1)) \cong V$ and 2) $H^{n+3}(\mathfrak{M}(n), \mathfrak{M}(n+1)) = 0$.

*1.* Relative cocycles of degree $n+2$ are of the form $(a, v+b)$ where $a, b \in \mathfrak{M}(n)$ (since there are no generators in degree $\leqslant 1$) and $v \in V$, such that $da = 0$ and $a = d(v+b) = dv + db$. Adding a coboundary $d(-b, 0)$ we obtain a normalised representative for the cohomology class $(a, v+b)$, which is $(a', v)$. This is only exact if $v = 0$. So sending $(a', v) \mapsto v$ is an injection. For surjectivity, any $v \in V$ is mapped to by the cocycle $(dv, v)$. Extend this by linearity to obtain a $k$-vector space isomorphism $H^{n+2}(\mathfrak{M}(n), \mathfrak{M}(n+1)) \cong V$. □

*2.* Since $\mathfrak{M}(n)$ and $\mathfrak{M}(n+1)$ are the same in all but degree $n+1$, this follows from the long-exact sequence for relative cohomology:

$$H^{n+2}(\mathfrak{M}(n)) \xrightarrow{\sim} H^{n+2}(\mathfrak{M}(n+1)) \xrightarrow{0} H^{n+3}(\mathfrak{M}(n), \mathfrak{M}(n+1)) \xrightarrow{0} H^{n+3}(\mathfrak{M}(n+1))$$

□

And finally, using the map of pairs $(id, \rho_{n+1}) : (\mathfrak{M}(n), \mathfrak{M}(n+1)) \to (\mathfrak{M}(n), \mathcal{A})$ we obtain from

$$\begin{array}{ccc} \mathfrak{M}(n) & \longrightarrow & \mathfrak{M}(n+1) \\ \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle \rho_{n+1}} \\ \mathfrak{M}(n) & \xrightarrow{\rho_n} & \mathcal{A} \end{array}$$

a morphism of long exact sequences

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^n(\mathfrak{M}(n+1)) & \longrightarrow & H^{n+1}(\mathfrak{M}(n), \mathfrak{M}(n+1)) & \longrightarrow & H^{n+1}(\mathfrak{M}(n)) & \longrightarrow & H^{n+1}(\mathfrak{M}(n+1)) \\ & \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle \clubsuit} & & \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle \diamondsuit} \\ \cdots \longrightarrow & H^n(\mathcal{A}) & \twoheadrightarrow & H^{n+1}(\mathfrak{M}(n), \mathcal{A}) & \xrightarrow{0} & H^{n+1}(\mathfrak{M}(n)) & \xhookrightarrow{\rho_n^*} & H^{n+1}(\mathcal{A}) \end{array}$$

$$\begin{array}{cccccccc} \longrightarrow H^{n+2}(\mathfrak{M}(n), \mathfrak{M}(n+1)) \cong V & \xrightarrow{0} & H^{n+2}(\mathfrak{M}(n)) & \xrightarrow{\sim} & H^{n+2}(\mathfrak{M}(n+1)) & \xrightarrow{0} & H^{n+3}(\mathfrak{M}(n), \mathfrak{M}(n+1)) = 0 \\ \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle \spadesuit} & & \big\downarrow{\scriptstyle 0} \\ \longrightarrow H^{n+2}(\mathfrak{M}(n), \mathcal{A}) = V & \xrightarrow{0} & H^{n+2}(\mathfrak{M}(n)) & \hookrightarrow & H^{n+2}(\mathcal{A}) & \longrightarrow & H^{n+3}(\mathfrak{M}(n), \mathcal{A}) \end{array}$$

where 0 comes from commutativity and induces the injectivity of the following map and the injectivity of ♠ comes from the commutativity of the left square, where the top map is surjective, and the left-bottom composite is injective. The surjectivity of ♣ comes from the left-bottom of the commutative square to the right, and the five lemma then gives the isomorphism ♢.

In particular, we have an isomorphism $H^*(\mathfrak{M}(n+1)) \to H^*(\mathcal{A})$ for $* \leqslant n+1$ and an injection for $* = n+2$.

Then, let $\mathfrak{M} = \varinjlim \mathfrak{M}(n)$ where $\rho : \mathfrak{M} \to \mathcal{A}$ is defined by $\rho|_{\mathfrak{M}(n)} = \rho_n$. Since cohomology commutes with filterd direct limits (this is an $\mathbb{N}$-filtered limit in an AB5 category), it follows that $\rho^* : H^*(\mathfrak{M}) \to H^*(\mathcal{A})$ is an isomorphism, and hence $(\mathfrak{M}, \mathcal{A})$ is a minimal model for $\mathcal{A}$.

## References

[GM13] Phillip Griffiths and John Morgan. *Rational homotopy theory and differential forms*, volume 16 of *Progress in Mathematics*. Springer, New York, second edition, 2013.