PRAMANA
— journal of
physics

pp. 1–20

The analytic structure of lattice models
Why can't we solve most models?

Anthony J. Guttmann
Department of Mathematics and Statistics,
University of Melbourne, Vic. 3010, Australia

Abstract. We investigate the solvability of a variety of well-known problems in lattice statistical mechanics. We provide a new numerical procedure which enables one to conjecture whether the solution falls into a class of functions called *differentially finite* functions. Almost all solved problems fall into this class. The fact that one can conjecture whether a given problem is or is not D-finite then informs one as to whether the solution is likely to be tractable or not. We also show how, for certain problems, it is possible to *prove* that the solutions are not D-finite, based on work by A. Rechnitzer[23–25].

Keywords. solvability, diferrentially ginite, bond animal, Ising model, susceptibility, self-avoiding walks, self-avoiding polygons

PACS Nos 05.50.+q, 02.90.+p

1. Introduction.

Some of the most famous results in mathematics prove intrinsic unsolvability of a particular problem. Examples include squaring the circle, and determining the roots of a general polynomial of degree > 4. In physics corresponding results include the impossibility of a perpetual motion machine.

Many problems in algebraic combinatorics and statistical mechanics can be formulated as lattice models. Very few can be exactly solved. Those that can include:
(i) The zero-field partition function of the two-dimensional Ising model [20]
(ii) The spontaneous magnetisation of the two-dimensional Ising model [28]
(iii) The square lattice dimer problem [18,9]
(iv) The six-vertex model [19]
(v) The eight-vertex model [1]
(vi) The hard hexagon model [2]

There are a number of simpler models which are also solvable, but most have strong geometric restrictions such as directedness and/or convexity. They include a range of polygon enumeration problems, enumerated by both perimeter and area, such as row-convex polygons, three-choice polygons, diagonally convex polygons and staircase polygons, see figure 1.

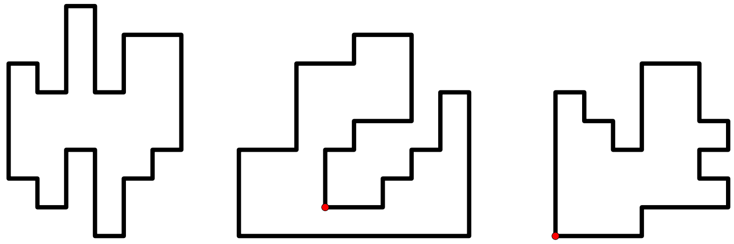

Figure 1. Some solvable polygon models. Reading from left to right are row-convex polygons, three-choice polygons and diagonally convex polygons.

There are also a number of solvable, constrained walk models, such as spiral walks, partially-directed self-avoiding walks and fully directed self-avoiding walks, as illustrated in figure 2.

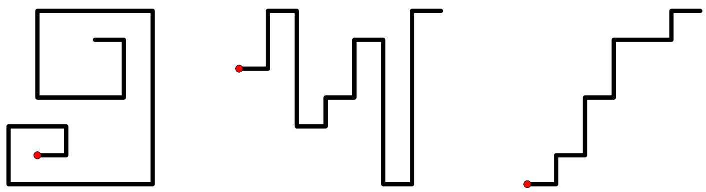

Figure 2. Some solvable walk models. Reading from left to right are spiral walks, partially-directed self-avoiding walks and fully directed self-avoiding walks.

Additionally there are several solvable animal enumeration problems, such as directed column-convex animals and directed animals on the square and triangular lattices.

There is a long list of celebrated unsolved problems in lattice statistical mechanics, prominent among which are:
(i) The susceptibility of the two-dimensional Ising model (semi-solved).
(ii) The partition function of the two-dimensional Ising model in a field.
(iii) Any property of the three-dimensional Ising model.
(iv) Two-dimensional self-avoiding walks and polygons.
(v) Two-dimensional percolation.
(vi) Two-dimensional directed percolation.
(vii) The q-state Potts model for $q > 2$.
(iix) Directed animals on the hexagonal lattice.

In seeking a solution, it is reasonable to ask what we mean by a solution? Ideally, one requires a (simple) closed form solution, such as that obtained by Onsager [20] or Yang [28] for the Ising model. Less restrictively, one may seek a difference or differential equation from which the properties of the solution may be extracted implicitly or explicitly. One might settle for a polynomial time algorithm to produce the coefficients in the series expansion of the solution.

Here we will to show how to *conjecture*, and then in some cases *prove* [23–25] that the above list of unsolved problems, as well as bond animals, bond trees, and directed bond animals cannot have solutions that belong to a certain large family called differentiably-finite functions.

A solution may usefully be considered a *generating function*, which is just a formal power series, in which the coefficients enumerate objects. For example, c_n may denote the number of objects of a certain type with n bonds. The generating function is $f(z) = \sum c_n x^n$. f may be a function of several variables, such as area and perimeter, or field and temperature.

Definition 1 *Let $f(x)$ be a formal power series in x.*
f is differentiably-finite, or D-finite if there exists a differential equation:

$$P_d(x)\frac{\partial^d f}{\partial x^d} + \ldots + P_1(x)\frac{\partial f}{\partial x} + P_0(x)f(x) = 0$$

where $P_k(x)$ are polynomials in x.

Note that the family of D-finite functions include algebraic and rational functions as a proper subset. The hierarchy of functions of a single variable we are considering here is illustrated in figure 3.

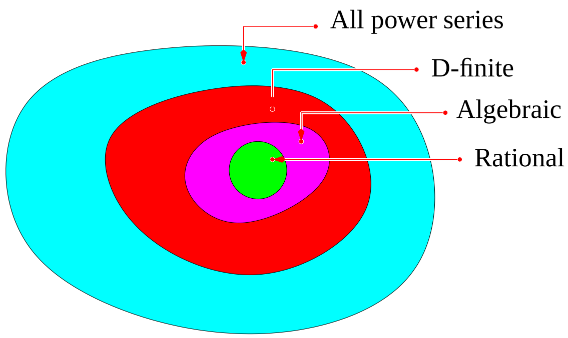

Figure 3. This figure shows the hierarchy of functions we are considering.

If these unsolved problems did have *D-finite* generating functions we could compute the coefficients of their generating function c_n in polynomial time. Almost all solved models, such as those listed above, have D-finite solutions. Based on this observation, a legitimate and effective method to try and solve such a problem is to generate (by counting) the first few terms of the power series, and then look for the underlying differential equation.

Note that most of the special functions of mathematical physics are particular cases of the hypergeometric functions, all of which are D-finite. The Gamma function is a notable exception. Most of the techniques we have for guessing generating functions look for D-finite functions, such as the computer programs *gfun* [10] and *newgrqd* [11]. Once one has a conjectured solution, it's much easier to prove it is correct.

The general approach to solving such problems is that if the object we are studying lives in $\mathbb{R}^3$ or $\mathbb{R}^2$, we simplify the model so that we can apply statistical mechanical and combinatorial techniques. For example, polymers in dilute solution live in $\mathbb{R}^3$. We wish to transform this to a combinatorial problem that one can embed in $\mathbb{Z}^3$. We illustrate this embedding in the two-dimensional case in figure 4.

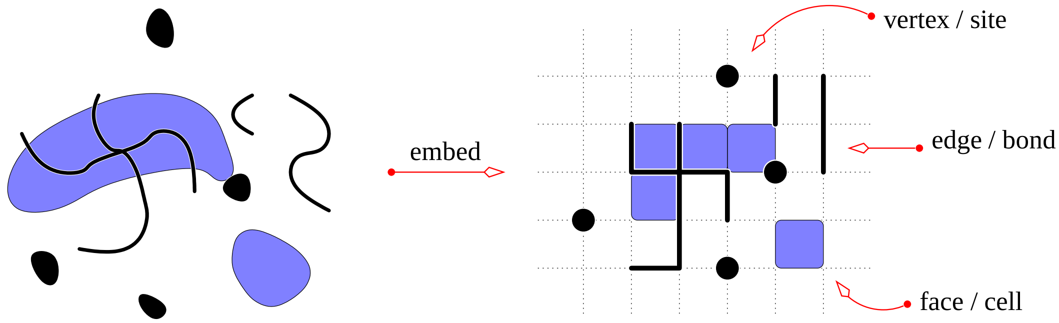

Figure 4. A general two-dimensional object in $\mathbb{R}^2$ embedded in $\mathbb{Z}^2$.

The two-dimensional object then becomes a picture on a grid made up of elementary pieces of a graph. These are vertices or sites, edges or bonds and faces or cells.

Let us illustrate some of these ideas by studying bond animals more carefully.

Definition 2 *A bond animal is: a finite connected union of bonds on the square lattice defined modulo translations.*

A typical bond animal on the square lattice is shown in figure 5.

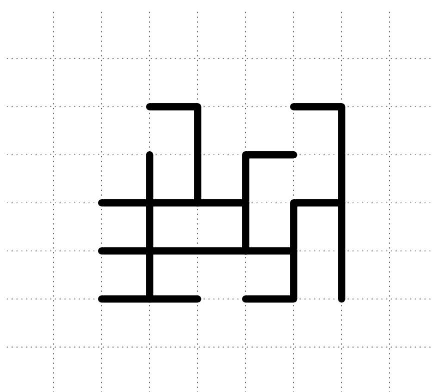

Figure 5. An example of a bond animal embedded on the square lattice.

Similar definitions hold for site animals and polyominoes. In this paper we will concentrate on bond animals, and the basic question we will ask is: How many bond animals can be constructed using n bonds? Denote this number c_n. Knowing c_n gives the partition function and so explains the physics of the problem. We illustrate the enumeration process below:

There are two ways of placing a single bond — $c_1 = 2$.

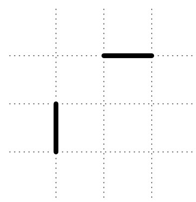

Figure 6. A single bond may be placed vertically or horizontally.

There are six ways of placing two bonds, so $c_2 = 6$

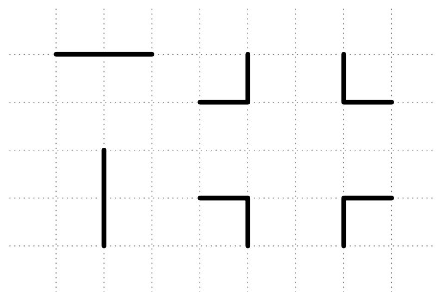

Figure 7. A two bond animal has six manifestations.

We can continue this by hand up to $n \approx 7$. However to progress further one must use a computer to count all animals. The number grows exponentially, that is to say, $c_n \sim A\lambda^n$. Hence the time taken to count animals grows as λ^n also, where typically $2 < \lambda < 10$, depending on the particular lattice. Any method that is quicker than this would be a good. Arguably a *nice* solution would be: a closed-form and *easily computable* expression for c_n, or a closed-form and *easily computable* expression for the ordinary or exponential generating function:

$$f(z) = \sum c_n z^n \qquad \text{or} \qquad g(z) = \sum c_n \frac{z^n}{n!}.$$

Less desirable, though still arguably a solution, is an algorithm or recurrence that computes c_n efficiently, preferably in polynomial time.

For example, consider counting rectangles (by perimeter), see figure 8. A solution of the form:

$$|R_n| = \sum_{r \in R_n} 1$$

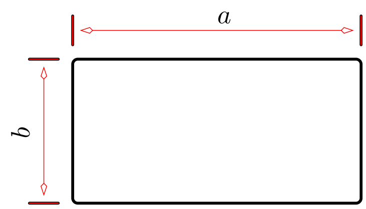

Figure 8. An example of a rectangle of dimension $a \times b$.

though true is not too helpful.

Whereas a solution of the form:

$$R(x) = \sum |R_n| x^n = \sum_{a \geq 1} \sum_{b \geq 1} x^{2a+2b} = \frac{x^4}{(1-x^2)^2}$$

is more informative. It can be inverted to show $|R_{2n}| = n - 1$

For most of the lattice models we are interested in, we can't do anything like this. Bond animals, bond trees, self-avoiding polygons, self-avoiding walks, and directed bond animals are all unsolved. These are illustrated in figure 9.

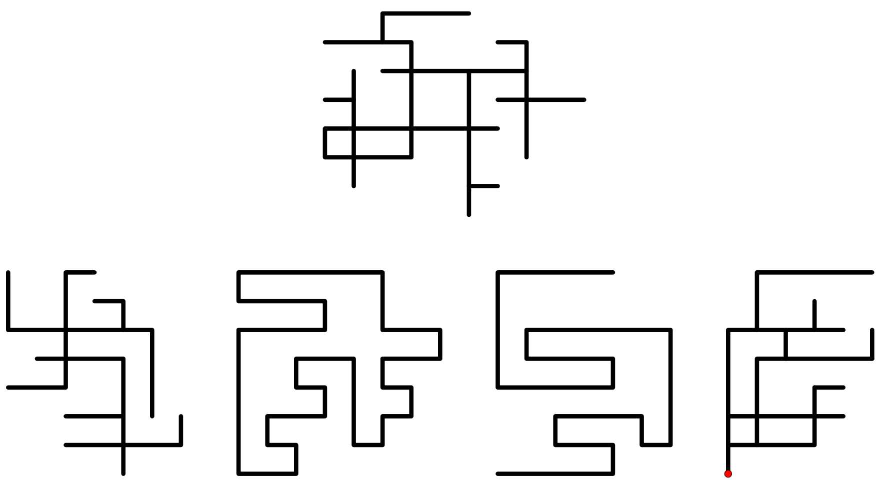

Figure 9. A variety of unsolved models. From left to right is shown a bond animal, a bond tree, a self-avoiding polygon, a self-avoiding walk and a directed bond animal.

Currently, the best solution for generating coefficients still takes exponential time (but less than λ^n.). This solution is based on the finite-lattice method [7,8,16]. It is exponential in time, but *exponentially faster* than brute force. Brute force for enumerating square lattice self-avoiding polygons takes time proportional to λ^n with $\lambda \approx 2.638$ Currently, the best algorithm for self-avoiding polygons (SAP) has complexity λ^n with $\lambda \approx 1.2$ ([17]). Similar improvements are achievable for other models. Despite much effort by many people over many years there are still no really nice solutions to the above problems. The two-dimensional Ising susceptibility, discussed in Section 3. below is a partial exception to this. Orrick, Nickel, Perk and Guttmann, [21,22] have developed a polynomial time algorithm for that problem. Almost as desirable would be algorithms that run in sub-exponential time, for example $\propto e^{\beta\sqrt{n}}$ such as Baxter's corner transfer matrix method, [5].

That is not to say that there is no other rigorous work. For these models it is known that the limit

$$\lim_{n\to\infty} \left(c_n\right)^{1/n} = \lambda$$

exists [14,15], and considerable effort and ingenuity is expended in finding rigorous bounds on λ for a variety of models.

There is also a large body of numerical and not-entirely-rigorous work that has greatly improved our understanding of the behaviour of these models.

In this paper we'll show how to conjecture, and outline how Rechnitzer [23–25] has proved, that bond animals, bond trees, self-avoiding polygons and directed bond animals cannot have D-finite solutions. Note that if these families did have D-finite generating functions we could compute c_n in polynomial time.

2. Proving non-D-finiteness

To prove that a given function is not D-finite, we first note that:

Theorem 1 *A D-finite power series has a finite number of singularities.*

For example the generating function of partitions of integers is

$$P(q) = \prod_{k\geq 1}(1-q^k)^{-1}.$$

It has singularities at $q = e^{2\pi i\theta}$ with $\theta \in \mathbb{Q}$. It is not D-finite.

Unfortunately, for the problems at hand we know almost nothing about the generating functions. We don't even know the location of the dominant singularity.

However by anisotropising the generating function we can prove something. The anisotropic generating function is:

$$P(x,y) = \sum_{n,m} c_{n,m} x^m y^n$$

where $c_{n,m}$ = the number of bond animals with m horizontal bonds and n vertical bonds, see figure 10.

The anisotropic generating function cuts the system up into an infinite sum of rational functions.

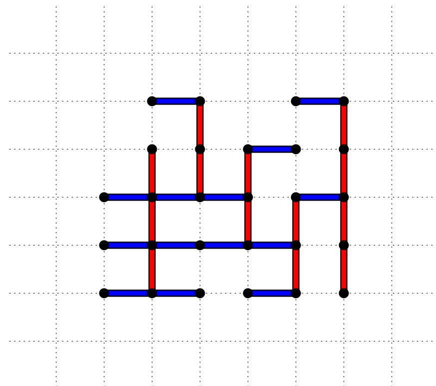

Figure 10. An example of a bond animal, with the horizontal and vertical bonds distinguished.

$$P(x,y) = \sum_{n,m} c_{n,m} x^m y^n = \sum_n H_n(x) y^n.$$

$H_n(x)$ is the generating function for objects with precisely n vertical bonds. It can be proved that it is a rational function. This follows from the fact that the coefficients of the generating function can be obtained from a finite-dimensional transfer matrix, from which the result follows [26]. We can compute the first few $H_n(x)$ exactly for most lattice problems. The purpose of this anotropisation is that the behaviour of these H_n functions reveal much about the solution. Let $P(x,y) = \sum H_n(x)y^n$ be an anisotropic generating function. For most families of bond animals, Rechnitzer has shown the $H_n(x)$:
(i) are rational functions of x,
(ii) have numerator degree less than or equal to the degree of the denominator,
(iii) have a denominator that is a product of cyclotomic polynomials.
The behaviour of the denominators is the key to the nature of the underlying, two-variable generating function.

Definition 3 *The cyclotomic polynomials, $\Psi_k(x)$, are the factors of the polynomials $(1-x^n)$. In particular, $(1-x^n) = \prod_{k|n} \Psi_k(x)$.*

$$\begin{aligned}
\Psi_1 &: (1-x) = \underline{(1-x)} \\
\Psi_2 &: (1-x^2) = (1-x)\ \underline{(1+x)} \\
\Psi_3 &: (1-x^3) = (1-x)\ \underline{(1+x+x^2)} \\
\Psi_4 &: (1-x^4) = (1-x)(1+x)\ \underline{(1+x^2)} \\
\Psi_5 &: (1-x^5) = (1-x)\ \underline{(1+x+x^2+x^3+x^4)} \\
\Psi_6 &: (1-x^6) = (1-x)(1+x)(1+x+x^2)\ \underline{(1-x+x^2)}
\end{aligned}$$

Let us look at a few of the solved models discussed above. Writing $H_n(x) = N_n(x)/D_n(x)$, we find that:

For directed walks $D_n(x) = (1-x)^{n+1}$.
For staircase polygons $D_n(x) = (1-x)^{2n+1}$.
For 3-choice polygons $D_n(x) = (1-x)^{a(n)}(1+x)^{b(n)}$, where $a(n)$ and $b(n)$ are known, linear functions of n, which depend on the parity of n [13].

3. The two-dimensional Ising model

For the two-dimensional Ising model partition function, the situation is even more remarkable. Denote the expansion variable in the x and y direction by $t_1 = \tanh(J_x/kT)$ and $t_2 = \tanh(J_y/kT)$ respectively. Then the logarithm of the reduced partition function can be written as

$$\log \Lambda(t_1, t_2) = \sum_{n,m} a_{n,m} t_1^{2m} t_2^{2n} = \sum_n R_n(t_1^2) t_2^{2n},$$

Baxter [3,4] showed that $R_n(t_1^2) = P_{2n-1}(t_1^2)/(1-t_1^2)^{2n-1}$.
That is, R_n is rational, with numerator and denominator being polynomials of degree $2n-1$, and with the denominator having a particularly simple structure. Observe that the only singularity in the complex t_1^2 plane is at $t_1^2 = 1$. Furthermore, there exists an inversion relation for the two-variable partition function,

$$\log \Lambda(t_1, t_2) + \log \Lambda(1/t_1, -t_2) = \log(1-t_2^2).$$

There is also the obvious symmetry relation

$$\Lambda(t_1, t_2) = \Lambda(t_2, t_1).$$

Remarkably, these two relations, plus the observed structure of R_n, that is to say, the property that R_n is a rational function with numerator and denominator polynomials of degree $2n-1$, is sufficient to determine, order by order, the numerator polynomials. This is because the $2n$ unknown numerator coefficients are reduced to n by the symmetry relation, while the inversion relation gives these n unknown coefficients. Alternatively expressed, *the complete Onsager solution is implicitly determined by the two functional relations, and the structure of* R_n. In practice, this means that, a mere 60 years after Onsager, we could have *conjectured* the exact solution from some simple numerical calculations—that of the first few R_ns. An attempt to do the same for the susceptibility fails [12] because the structure of the R_ns is not so simple. Unsolved models, such as the Ising susceptibility and others listed above, are very different.

3.1 Susceptibility.

For the susceptibility there is a similar inversion relation

$$\chi(t_1, t_2) + \chi(1/t_1, -t_2) = 0$$

as well as the symmetry relation $\chi(t_1, t_2) = \chi(t_2, t_1)$. However the denominator polynomials, the first few of which are shown below, do not display the nice property, displayed by the corresponding free-energy polynomials, that their degree is $2n - 1$ [12].

$$
\begin{aligned}
D_1(x) &= (1-x) \\
D_2(x) &= (1-x)^2 \\
D_3(x) &= (1-x)^3(1+x) \\
D_4(x) &= (1-x)^4 \\
D_5(x) &= (1-x)^5(1+x)^4(1+x+x^2) \\
D_6(x) &= (1-x)^6(1+x)^2 \\
D_7(x) &= (1-x)^7(1+x)^5(1+x+x^2)^3
\end{aligned}
$$

It can be seen that the denominator polynomials are not of degree $(2n - 1)$, but larger, and display the property that they are products of cylotomic polynomials of ever increasing degree. Hence the symmetry and inversion relations are not sufficient to identify the unknown numerator coefficients.

For the two-dimensional Ising model susceptibility, Orrick, Nickel, Guttmann and Perk [21,22] provided a polynomial time algorithm for the coefficients, used it to generate more than 300 series coefficients, and then analyzed the scaling behaviour from the generating function.

3.2 Self-avoiding polygons.

For self-avoiding polygons on the square lattice the denominator polynomials also display the property that they are products of cylotomic polynomials of ever increasing degree. We have calculated the first few, and they are:

$$
\begin{aligned}
D_1(x) &= (1-x) \\
D_2(x) &= (1-x)^3 \\
D_3(x) &= (1-x)^5 \\
D_4(x) &= (1-x)^7 \\
D_5(x) &= (1-x)^9(1+x)^2 \\
D_6(x) &= (1-x)^{11}(1+x)^4 \\
D_7(x) &= (1-x)^{13}(1+x)^6(1+x+x^2) \\
D_8(x) &= (1-x)^{15}(1+x)^8(1+x+x^2)^3 \\
D_9(x) &= (1-x)^{17}(1+x)^{10}(1+x+x^2)^5 \\
D_{10}(x) &= (1-x)^{19}(1+x)^{12}(1+x+x^2)^7(1+x^2).
\end{aligned}
$$

3.3 The general case.

More generally, Guttmann and Enting [12] observed that, almost all solved models have simple denominators, such as those of the Ising model partition function, or staircase polygons, while unsolved models have poles that *become dense on* $|x| = 1$. They suggested this as a test of solvability. This notion of solvability can be made more precise. A D-finite power series cannot have this dense pole denominator pattern:

Theorem 2 (Bousquet-Mélou)
Let $P(x, y) = \sum H_n(x)y^n$ be a D-finite power series, and let $\mathcal{S}$ be the set of all singularities of the $H_n(x)$. The set $\mathcal{S}$ has a finite number of accumulation points.

For the unsolved models we have investigated numerically, the build-up of denominator zeros leads us to *conjecture* that they become dense on the unit circle. Hence their anisotropic generating function cannot be D-finite. Rechnitzer [23–25] has proved that:

Theorem 3 *The anisotropic generating functions of bond animals, bond trees, self-avoiding polygons and directed bond animals have singularities that form a dense set on the unit circle. Hence these functions are not D-finite.*

A secondary result is:

Theorem 4 *For most families of bond animals, the denominators of $H_n(x)$ are bounded in the sense that $D_n(x)$ is a divisor of*

$$\Psi_1(x)^{3n+1} \prod_{k=2}^{\lfloor n/2 \rfloor + 1} \Psi_k(x)^{2n-3k+4}.$$

For self-avoiding polygons $D_n(x)$ is a divisor of:

$$\prod_{k=1}^{\lceil n/3 \rceil} \Psi_k(x)^{2n-6k+5}.$$

These bounds appear to be very tight: For bond animals and bond trees the bound is (probably) exact. For SAPs the bound is out by a single factor of $\Psi_2(x) = (1 + x)$

This determines half of the unknown anisotropic generating function:

$$P(x, y) = \sum_n \frac{N_n(x)}{D_n(x)} \, y^n$$

We know $D_n(x)$ — for a solution we just need to find $N_n(x)$.

Unfortunately the numerators are much harder, and are usually obtained by direct construction from the known enumerations. These theorems above are obtained by squashing animals or what Rechnitzer [23] calls haruspicy. The dictionary definition of haruspicy is telling the future by examining the entrails of animals. In the next section we outline Rechnitzer's proof of the above theorems.

4. Proofs by haruspicy.

Consider animals with 0 vertical bonds, $\mathcal{G}_{\prime}$:

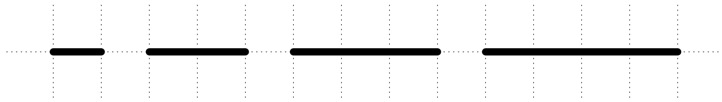

Figure 11. The first four animals with 0 vertical bonds.

It is clear that the generating function for such animals is $H_0(x) = \frac{1}{1-x}$. Any animal in this set can be squashed down to the single bond.

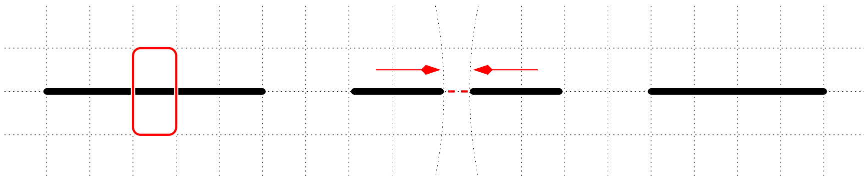

Figure 12. Showing bond deletion or "squashing" of an animal.

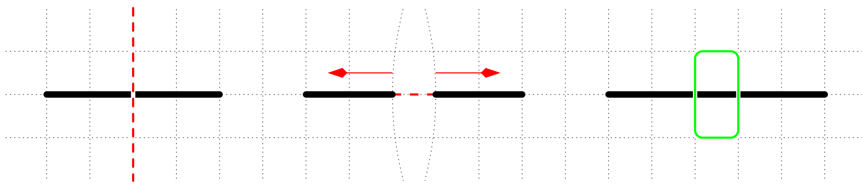

Figure 13. Showing bond insertion or "unsquashing" of an animal.

But we can also reverse this. The single horizontal bond is minimal under squashing. The other animals in $\mathcal{G}_{\prime}$ come from it by unsquashing. More generally we cannot squash/expand bonds as we like. We have to preserve the "topology" of the configurations. That is, we must preserve self-avoidance.

Definition 4 *A column of an animal is characterised by the horizontal bonds within a horizontal lattice spacing. If there are k horizontal bonds in the column, we call it a k-column.*

In figure 14 we show an animal with a 4-column. When two neighbouring columns are duplicates, we are able to remove one of them.

An animal that cannot be squashed further is a minimal animal. $\mathcal{G}_{\prime}$ has 1 minimal animal. $\mathcal{G}_{\infty}$ has lots (the number is growing fast):

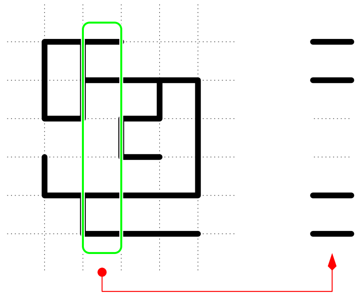

Figure 14. Showing a bond animal with a (highlighted) 4-column.

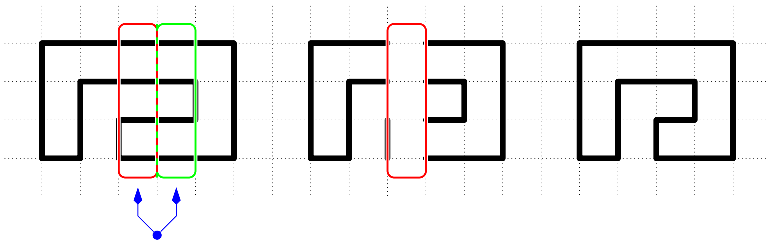

Figure 15. Showing the operation of deleting a 3-column.

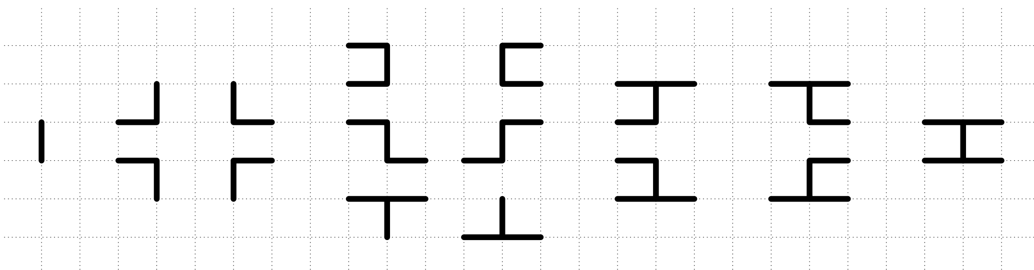

Figure 16. The sixteen minimal animals in $\mathcal{G}_\infty$.

The key point to bear in mind is that $\mathcal{G}_\backslash$ has a finite number of minimal animals (but an *infinite* number of non-minimal animals).

It is not possible to reduce an animal to 2 different minimal animals. This gives an equivalence relation. Two animals A and B are equivalent if they reduce to the same minimal animal.

The equivalence relation partitions $\mathcal{G}_\backslash$ into equivalence classes: The minimal animal contains all the information needed to produce the generating function of the equivalence class.

One can squash all members of an equivalence class into a minimal animal, or expand a

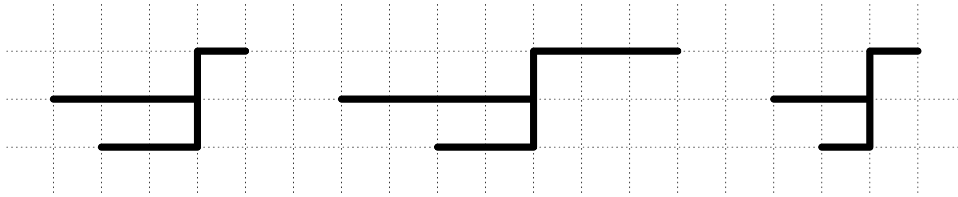

Figure 17. All these animals reduce to the same minimal animal in $\mathcal{G}_{\in}$.

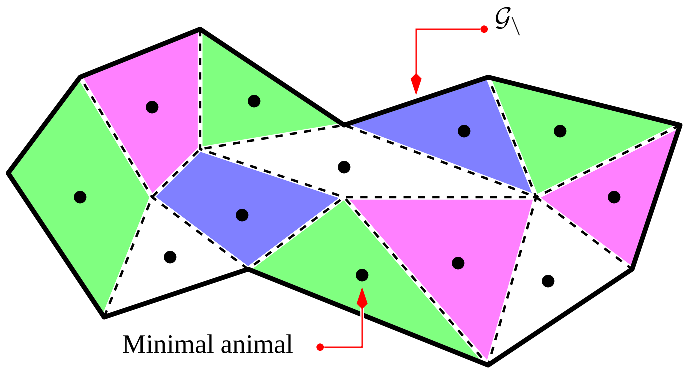

Figure 18. An illustration of the equivalence class partitioning of $\mathcal{G}_{\backslash}$.

minimal animal into the generating function for that equivalence class. This is illustrated i in figure 19.

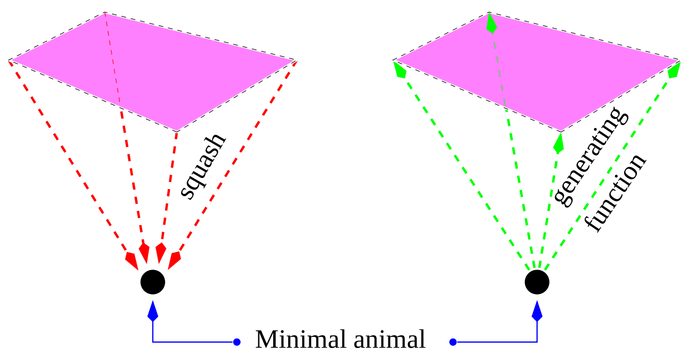

Figure 19. An illustration of the equivalence class partititoning of $\mathcal{G}_{\backslash}$.

Lemma 1 *Let A be a minimal animal. Let $\gamma_k(A)$ be the number of k-columns in A. The generating function of all animals in the equivalence class of A is:*

$$GF(A) = \prod_k \left(\frac{x^k}{1-x^k} \right)^{\gamma_k(A)}$$

The way these factors in the product above arise can be seen from the figure 20.

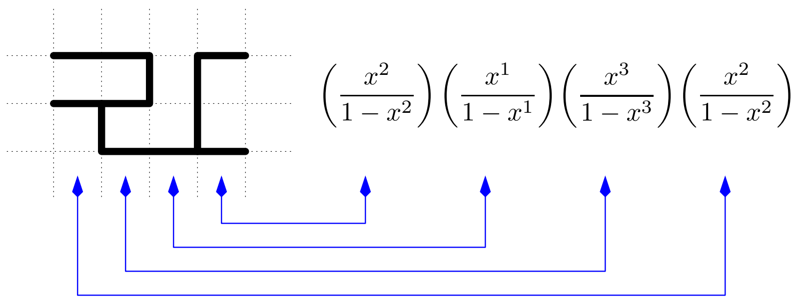

Figure 20. An illustration of the mapping between k-columns and the factors in the generating function.

We can now write $H_n(x)$ as a sum over the minimal animals in $\mathcal{G}_\backslash$. Each generating function is of the above form. Hence

$$H_n(x) = \sum_{A \in \mathcal{M}_\backslash} \prod_k \left(\frac{x^k}{1-x^k} \right)^{\gamma_k(A)}$$

For any given n, the number of summands is finite, so we have:

Theorem 5 *If $P(x,y) = \sum_{n \geq 0} H_n(x) y^n$ is the anisotropic generating function of a set of animals, $\mathcal{G}$, then $H_n(x)$ is a rational function, such that the degree of the numerator is less than the degree of the denominator, and the denominator is a product of cyclotomic polynomials.*

We can also see which singularities are caused by which minimal animals:

Theorem 6 *If the denominator of $H_n(x)$ contains a cyclotomic factor $\Psi_k(x)$, then there exists a minimal animal A in $\mathcal{G}_\backslash$ such that A contains a k-column.*

If it takes at least V vertical bonds to build a bond animal with a k-column then

$$\Psi_k(x) \not| D_n(x) \qquad \text{for } n < V.$$

Similarly multiple k-columns give rise to multiple factors of $\Psi_k(x)$ and *vice versa*. This is the idea behind the proof of the denominator bounds. More precisely, one needs to answer the questions:
(i) When can there be a factor of $\Psi_k^\alpha(x)$?
(ii) How many vertical bonds do you need to make α k-columns?

It is complicated by the fact that we need a finer equivalence relation than just column-deletion. This is because the horizontal bonds in many configurations can be squashed and expanded independently of each other, see figure 21.

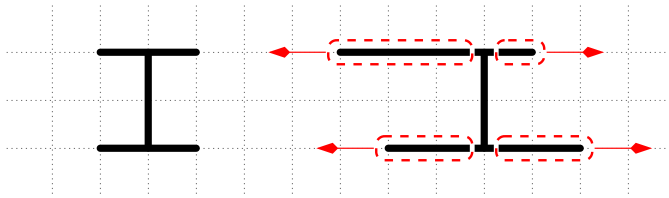

Figure 21. An illustration of the possibility of independent expansion of bonds in a minimal animal.

To prove non-D-finiteness we study those animals that cause the first occurrence of $\Psi_k(x)$. These seem to have simple patterns: For SAPs new cyclotomic polynomials $\Psi_k(x)$ enters in $H_{3k-2}(x)$

∘ these poles are caused by so-called *2-4-2*-polygons.

For the others new $\Psi_k(x)$ enters in $H_{2k-2}(x)$.

∘ these poles are caused by *2*-animals.

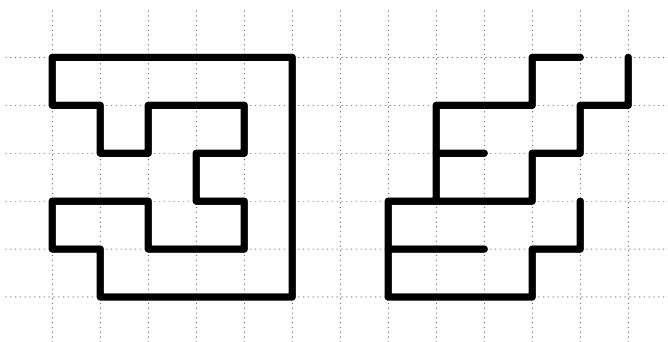

Figure 22. A $2-4-2$ polygon (left) and a 2-animal (right).

The point is that, in both cases, horizontal bonds cannot be arbitrarily unsquashed, or the self-avoidance constraint will be violated.

Bond animal denominators contain higher order cylotomic polynomials caused by *2*-animals, as seen in figure 23.

SAP denominators contain higher order cyclotomic polynomials caused by *2-4-2*-polygons. These are so-called as the minimum polygon contains, alternately, a 2-column, a 4-column and a 2-column.

This allows observation allows us to split the $H_n(x)$ into two parts:

$$H_n(x) = \underbrace{A_n(x)}_{\text{no } \Psi_k(x)} + \underbrace{B_n(x)}_{\text{has } \Psi_k(x)}$$

$$D_0(x) = (1-x)$$

$$D_1(x) = (1-x)^4$$

$$D_2(x) = (1-x)^7(1+x)^2$$

$$D_3(x) = (1-x)^{10}(1+x)^4$$

$$D_4(x) = (1-x)^{13}(1+x)^6(1+x+x^2)^3$$

$$D_5(x) = (1-x)^{16}(1+x)^8(1+x+x^2)^5$$

$$D_6(x) = (1-x)^{19}(1+x)^{10}(1+x+x^2)^7(1+x^2)^4$$

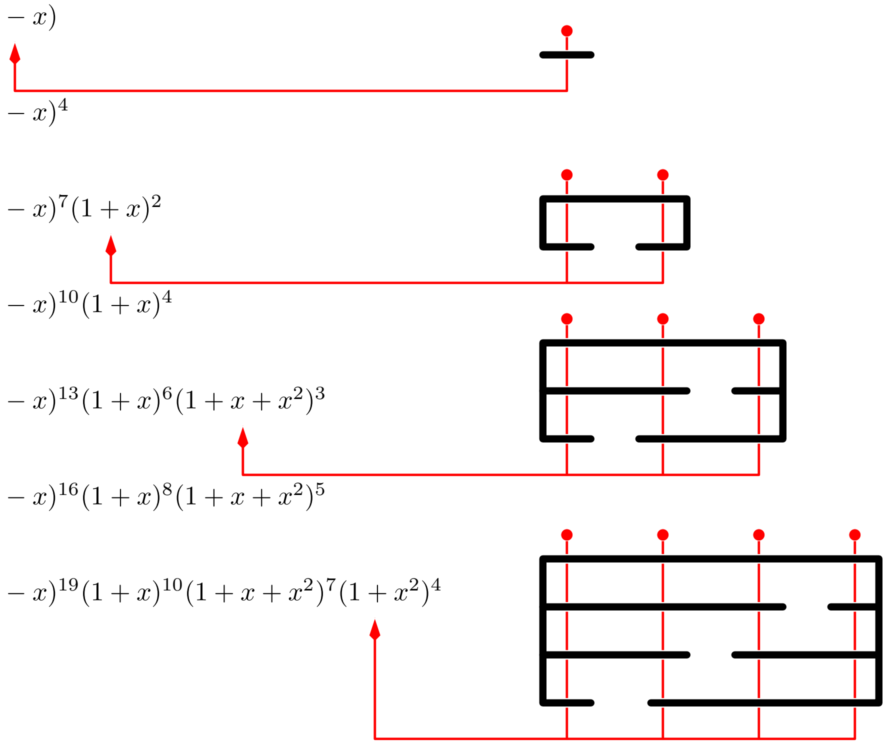

Figure 23. 2-animals giving rise to higher order cyclotomic polynomials.

For SAPs this becomes:

$$H_n(x) = \underbrace{A_n(x)}_{\text{not } 2\text{-}4\text{-}2} + \underbrace{B_n(x)}_{\text{is } 2\text{-}4\text{-}2}$$

When we add A_n and B_n all the Ψ_k's come from B_n. If there are cancellations in the $H_n(x)$ they come from B_n. We cannot compute $A_n(x)$, however we can compute $B_n(x)$. For SAPs this means counting all *2-4-2*-polygons. For the others it means counting certain *2*-animals. To do this we use the Temperley method [27]. In this methodology, one builds animals row-by-row [27,6].

This enumeration gives rise to the functional equations such as:

$$f(s;x,y) = \frac{xys}{1-xs} + \sum_{k=0}^{5} c_{k+1} \left(\frac{\partial}{\partial s}\right)^k f(s;x,y)\Big|_{s=1} + c_7 f(s,x,y) + c_8 f(sx;x,y),$$

where $f(s;x,y)$ is the generating function of *2-4-2*-polygons and the c_k are rational functions of s, x and y. Their expressions are not nice, and involve many pages of ones favourite computer algebra language.

$$D_1(x) = (1 - x)$$

$$D_2(x) = (1 - x)^3$$

$$D_3(x) = (1 - x)^5$$

$$D_4(x) = (1 - x)^7(1 + x)$$

$$D_5(x) = (1 - x)^9(1 + x)^3$$

$$D_6(x) = (1 - x)^{11}(1 + x)^5$$

$$D_7(x) = (1 - x)^{13}(1 + x)^7(1 + x + x^2)$$

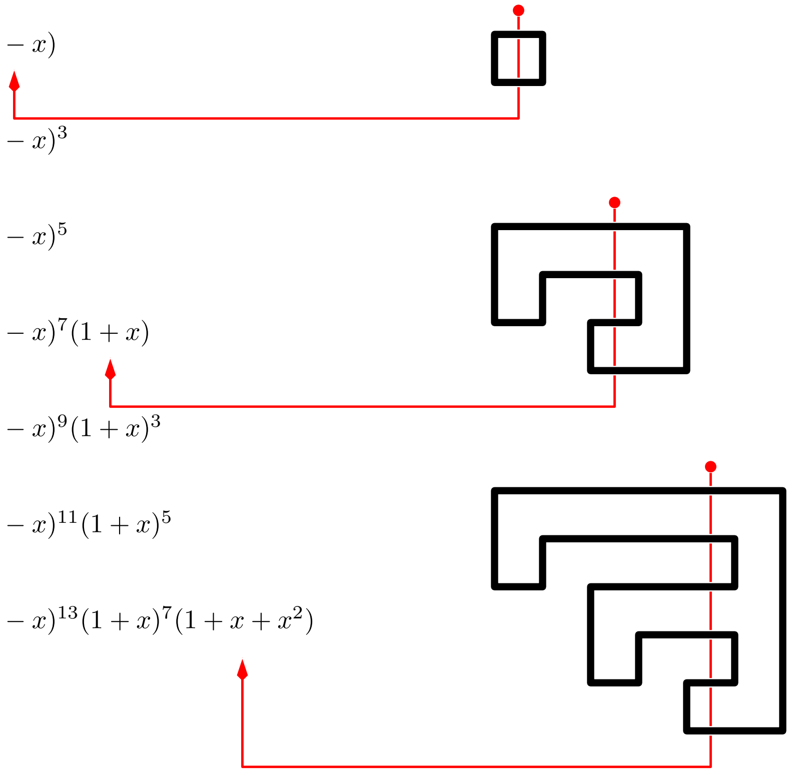

Figure 24. $2 - 4 - 2$ polygons giving rise to higher order cyclotomic polynomials.

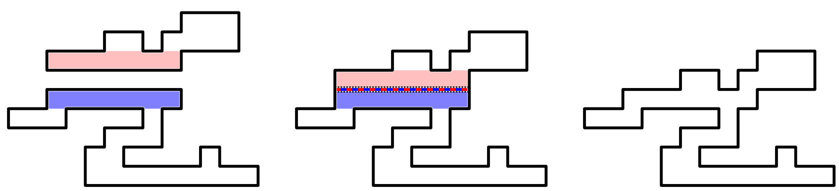

Figure 25. Illustration of the Temperley method to build a $2 - 4 - 2$ polygon and then squashing it by elimination of a duplicate 2-row.

Fortunately, the only term that is important is the last one, which is tractable. That term, "$c_8 f(sx; x, y)$," is the only one in the functional equation that introduces new singularities, through the following mechanism, which we illustrate schematically:

$$\frac{\langle \text{ some polynomial } \rangle}{1 - sx^k} \xrightarrow[\text{iterate}]{s = sx} \frac{\langle \text{ some other polynomial } \rangle}{1 - sx^{k+1}}$$

This operation takes factors of $(1 - sx^k)$ and makes new factors of $(1 - sx^{k+1})$. Analysis of this term then shows that new $\Psi_k(x)$ appear where we expect them to. That is to say, for SAPs, 1 factor of $\Psi_k(x)$ appears in $H_{3k-2}(x)$. For animals, factors of $\Psi_k(x)$ appear in $H_{2k-2}(x)$. Additionally, Rechnitzer [23] can prove that there are no cancellations between the numerator and denominator. For SAPs, $H_{3k-2}(x)$ has singularities at $\Psi_k(x) = 0$. while for bond animals, $H_{2k-2}(x)$ has singularities at $\Psi_k(x) = 0$. The set of all singularities is dense on $|x| = 1$. Therefore, by the above theorem, the anisotropic generating functions are not D-finite. QED.

5. Discussion and conclusion

We have shown how to *conjecture* and in some cases *prove* that a variety of well-known problems in algebraic combinatorics and statistical mechanics have solutions that are not D-finite.

Current and future work in this direction includes trying to prove non-D-finiteness for self-avoiding walks, the partition function and other thermodynamic functions of the q-state Potts model, site animals and polyominoes.

Other future important research directions include the questions: Can it help with series expansion techniques? Can it tell us anything about the numerators? Even partial answers to these questions would be extremely valuable.

Acknowledgements

I would like to thank Andrew Rechnitzer for help both with the figures and with the idiosyncrasies of Latex style files. I gratefully acknowledge financial support from the Australian Research Council.

References

[1] R. J. Baxter, Phys. Rev. Lett., **26**, 834-7, (1971).
[2] R. J. Baxter, J. Phys. A. **13**, (1980) L61-70.
[3] R. J. Baxter, in *Fundamental Problems in Statistical Mechanics*, **5**, (1981) 109-41. E. G. D. Cohen ed. (North Holl. Amsterdam).
[4] R. J. Baxter, J. Stat. Phys, **28**, (1982) 1-41.
[5] R. J. Baxter, *Exactly solved models in statistical mechanics.*, Academic Press, London, (1982).
[6] M. Bousquet-Mélou, Disc. Math. **154**, (1996) 1-25.
[7] T. de Neef and I. G. Enting, J. Phys. A: Math. Gen. **10**, (1977) 801-5.
[8] I. G. Enting, Nucl. Phys. B (Proc. Suppl.) **47**, (1996) 180-7.
[9] M. E. Fisher, Phys. Rev., **124**, 1644-72, (1961).
[10] GFUN, a program developed by B. Salvy, P. Zimmerman, F. Chyzak and colleagues at INRIA, France. Available from http://pauillac.inria.fr/algo
[11] A. J. Guttmann, Asymptotic Analysis of Power Series Expansions, in *Phase Transitions and Critical Phenomena*, **13**, eds. C. Domb and J. Lebowitz, Academic Press, (1989) 1-234.
[12] A. J. Guttmann and I. G. Enting, Phys. Rev. Letts. **76**, (1996), 344-7.
[13] A. J. Guttmann and I. Jensen, (in preparation).
[14] J. M. Hammersley, Proc. Camb. Phil. Soc. **57** 516-523, (1961).

[15] J. M. Hammersley and D. J. A. Welsh, Q. J. Math. (Oxford, 2nd series), **13**, 108-10, (1962).
[16] I. Jensen and A. J. Guttmann, J. Phys. A:Math. Gen, **32**, 4867-76, (1999).
[17] I. Jensen, J. Phys. A:Math. Gen, **37**, 5503-24, (2004).
[18] P. W. Kasteleyn, Physica, **27**, 1209-25, (1961).
[19] E. H. Lieb, Phys. Rev. Lett. **18**, 1046-8, (1967).
[20] L Onsager, Phys. Rev. **65** (1944) 117-49.
[21] W. P. Orrick, B. Nickel, A. J. Guttmann, J. H. H. Perk, Phys. Rev. Lett, **86**, 4120-3, (2001).
[22] W. P. Orrick, B. Nickel, A. J. Guttmann, J. H. H. Perk, J. Stat. Phys, **102**, 795-841, (2001).
[23] A. Rechnitzer, Haruspicy and anisotropic generating functions. Advances in Applied Mathematics, **30**, 228-257, (2003).
[24] A. Rechnitzer. Haruspicy 2: The self-avoiding polygon generating function is not D-finite. Accepted for publication in Journal of Combinatorial Theory - Series A.
[25] A. Rechnitzer. Haruspicy 3: The directed bond-animal generating function is not D-finite. To appear in Journal of Combinatorial Theory - Series A.
[26] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, (1999).
[27] H. N. V. Temperley, Phys. Rev., **103** 1-16, (1956)
[28] C. N. Yang, Phys. Rev. **85** 808-16 (1952)