Lattice paths: vicious walkers and friendly walkers

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1 Introduction

In an earlier paper [4] the problem of vicious random walkers on a d-dimensional directed lattice was considered. "Vicious walkers" describes the situation in which two or more walkers arriving at the same lattice site annihilate one another. Accordingly, the only allowed configurations are those in which contacts are forbidden. Alternatively expressed as a static rather than dynamic problem, vicious walkers are mutually self-avoiding networks of directed lattice walks, that is directed lattice paths, which in turn model directed polymer networks.

The problem of vicious walkers was introduced to the mathematical physics literature by Fisher [5] in 1984, who also discussed a number of physical applications. The general model is one of P random walkers on a d-dimensional lattice, who at regular time intervals simultaneously take one step with equal probability in the direction of one of the allowed lattice vectors. In the combinatorics literature, an equivalent problem for any planar graph was treated by Gessel and Viennot [7], though some of their results had been anticipated in other settings [16, 15].

In considering lattice path problems in which the Gessel-Viennot formulation does not apply, Viennot [23] suggested a model where the mutual avoidance constraint is relaxed to the extent that two paths may share a site, and may even stay together for just one step, but must then diverge – though they may subsequently touch. We now call this model "the 2-friendly-walk model" since the number of consecutive sites the walkers are allowed to share, here two, distinguishes this model from vicious walkers, which are not allowed to share any consecutive sites. It also distinguishes it from the so-called osculating walkers, which form directed lattice paths that can share one, but no more than one, consecutive sites. This naming convention also leads us to make further natural generalisations, given below.

Interestingly, each of the vicious, osculating and 2-friendly-walk models can be formulated as lattice statistical mechanical vertex models, which are models of ferroelectric materials (see

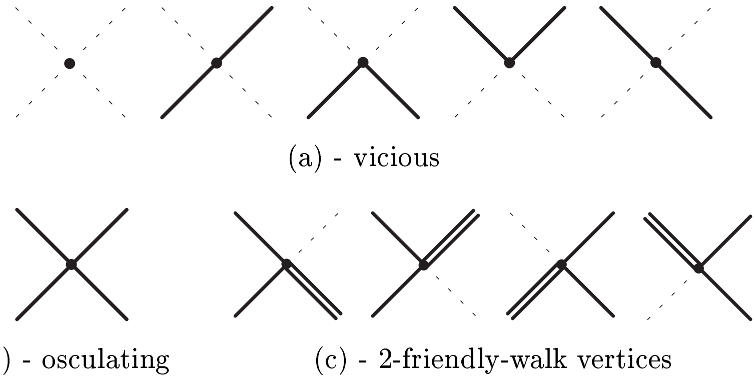


Figure 1: Part (a) shows the vertices of the vicious walker model. Figure 1(b) shows the additional vertex that occurs in the osculating walker model. Figure 1(c) shows the extra 2-friendly-walk vertices.

[13]). In particular the vicious, osculating and 2-friendly-walk models can be mapped to five, six and ten vertex models, respectively, as shown in figure 1.

We now make the natural generalisation of the vicious, osculating and 2-friendly-walk models to models where the walks are allowed to share up to n consecutive sites. These models are then referred to as the "n-friendly-walk models". It is important to note that in any of the models considered in this paper, only two walkers are allowed to share any site. In our naming scheme the vicious and osculating walker models become the 0- and 1-friendly-walk models respectively. One may think of each of the models of increasing n, starting at the vicious walkers, moving through osculating, and onwards, as allowing the walkers to become increasing friendly. Hence we shall refer to the n-friendly-walk models other than vicious walkers as partially-friendly walkers, the degree of friendliness being labelled by n.

In this paper we shall consider the n-friendly-walk model on the square, and more generally on the directed d-dimensional body-centred-hyper-cubic lattices. To define the n-friendly-walk model on the directed square lattice rotated through 45°, so that the unit vectors on the lattice are $(\mathbf{i}+\mathbf{j})/\sqrt{2}$ and $(\mathbf{i}-\mathbf{j})/\sqrt{2}$, let us consider P lattice paths labelled as $p = 1, \ldots, P$ of length L (see figure 2) which start at $l = 0$ at vertical positions $z_{p,0}$ having intermediate positions $z_{p,l}$ after l steps and finishing at $l = L$ at $z_{p,L}$. These P walks never cross so that they are always ordered with $z_{p,l} \leq z_{p',l}$ for $p < p'$. The conditions for the n-friendly-walk model can then be expressed as follows:

(1) All paths must fulfill the *non-crossing condition*, that is $z_{p,l} \leq z_{p+1,l}$ for $1 \leq p \leq P-1$, $0 \leq l \leq L$.

(2) They must satisfy the *n-consecutive-site* or *n-friendly condition.* That is two walkers may stay together for at most n consecutive sites For $0 \leq l \leq L-n$, $1 \leq p \leq P-1$, there exists at least one $k \in [0, n]$ such that $z_{p,l+k} \neq z_{p+1,l+k}$.

(3) One must also exclude three walkers occupying the same site: the *no-three-walker con-*

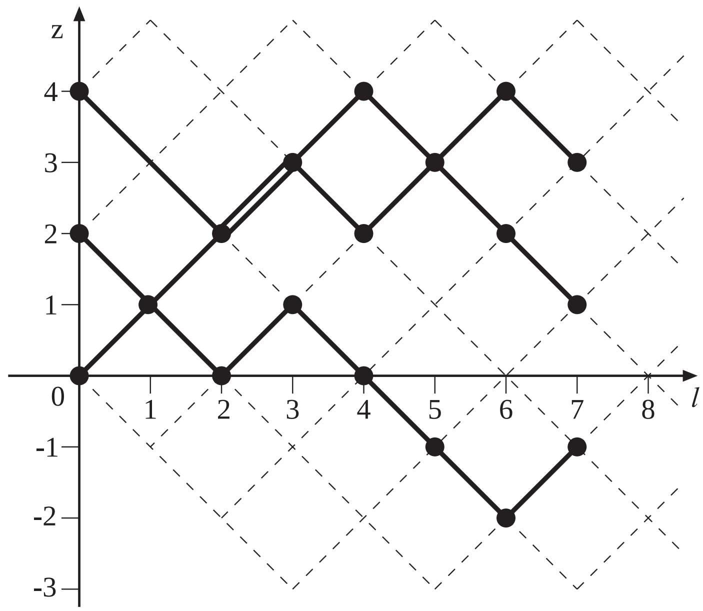


Figure 2: A watermelon of length $L = 7$ with three walkers ($P = 3$) in the 2-friendly-walk model on the rotated directed square lattice. Note that at $(1, 1)$ and $(5, 3)$ two walkers share a site and further the two upper walkers share the two consecutive sites $(2, 2)$ and $(3, 3)$.

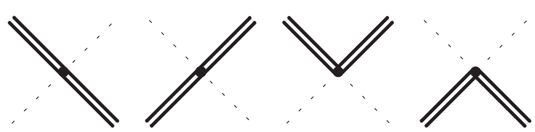


Figure 3: The four vertices in the ∞-friendly-walk model that occur in addition to the ten of the 2-friendly-walk model.

dition. This is achieved by requiring $z_{p,l} < z_{p+2,l}$ for $1 \leq p \leq P - 2$ and $0 \leq l \leq L$.

Note that for $m \leq n$ any m-friendly-walk configuration is also a valid n-friendly-walk configuration, so by increasing n one increases the size of the configuration space of the model.

We consider in this paper one further model, which for convenience we shall call the ∞-friendly-walk model. In this model two adjacent walks may share any number of sites; that is, we simply remove the condition (2) above from our definition of n-friendly-walks. Intriguingly, the ∞-friendly-walk model, in contrast to the n-friendly-walk models with $n \geq 3$, though in common with the n-friendly-walk models with $n = 0, 1, 2$, as described above, can be formulated as a vertex model with fourteen vertices. The four vertices that occur in addition to those of the 2-friendly-walk model are shown in figure 3. We point out that the ∞-friendly-walk model differs by virtue of the "no-three-walker condition"(3) from a model where walks are directed and stay ordered vertically at each time step but have any number of walks occupying the same site. This latter model, in which only condition (1) above is obeyed, may simply be called the 'non-crossing walker model'. We also note in passing that by allowing more than two walkers at one site but keeping the restriction of two walkers per bond, one obtains a walk model that can be mapped to a nineteen vertex model.

Two standard topologies of interest for networks of lattice paths are that of a *star* and a *watermelon*. Consider again a directed square lattice rotated 45°. Both configurations consist of P chains of length L which start at $(0,0),(0,2),(0,4),\cdots,(0,2P-2)$. Watermelon configurations end at $(L,k),(L,k+2),(L,k+4),\cdots,(L,k+2P-2)$. For stars, the end-points of the chain all have l-coordinate equal to L, but the z-coordinates are unconstrained, apart from the ordering imposed by the ordering of the walks. Thus if the end-points are $(L,z_{1,L}),(L,z_{2,L}),\cdots,(L,z_{P,L})$, then $-L \leq z_{1,L} \leq z_{2,L} \leq \cdots \leq z_{P,L} \leq 2P-2+L$, with $z_{p,l}$ denoting the z-coordinate of the p-th walker after the l-th step. Note that for vicious walkers these inequalities would be strict (apart from the first and the last). We shall use $S_{P,\mathrm{model}}$ and $W_{P,\mathrm{model}}$ to denote the generating functions for stars and watermelons respectively where the subscript "model" is given in this paper by either the integer n to refer to the n-friendly-walk model or the symbol ∞ to refer to the ∞-friendly-walk model. In particular we will consider generating functions for both isotropic and anisotropic models on the directed, rotated square lattice. In the first case we denote them $S_{P,\mathrm{model}}(t)$ and $W_{P,\mathrm{model}}(t)$, where t is the variable associated with the length of the walks. Hence

$$S_{P,\mathrm{model}}(t) = \sum_{L=0}^{\infty} s_{P,\mathrm{model},L}\, t^L \tag{1.1}$$

and

$$W_{P,\mathrm{model}}(t) = \sum_{L=0}^{\infty} w_{P,\mathrm{model},L}\, t^L \tag{1.2}$$

where $s_{P,\mathrm{model},L}$ and $w_{P,\mathrm{model},L}$ are the numbers of valid star and watermelon configurations respectively of P walks of length L in the model described by the associated subscript. In the anisotropic case we distinguish between NE and SE steps. In that case we write the generating functions as $S_{P,\mathrm{model}}(x,y)$ and $W_{P,\mathrm{model}}(x,y)$ where x (y) is associated with the number of NE (SE) steps in the graphs. Hence

$$S_{P,\mathrm{model}}(x,y) = \sum_{j,k=0}^{\infty} s_{P,\mathrm{model},j,k}\, x^j y^k \tag{1.3}$$

and

$$W_{P,\mathrm{model}}(x,y) = \sum_{j,k=0}^{\infty} w_{P,\mathrm{model},j,k}\, x^j y^k \tag{1.4}$$

where $s_{P,\mathrm{model},j,k}$ and $w_{P,\mathrm{model},j,k}$ are the numbers of valid star and watermelon configurations respectively of P walks with j NE steps and k SE steps in total in the model described by the associated subscript. Note that we associate the variables in the anisotropic generating function with the steps in the graph and in the isotropic generating function we associate the variable with the length of the walks. Hence $s_{P,\text{model},j,k} \neq 0$ and $w_{P,\text{model},j,k} \neq 0$ only if $j + k = 0 \pmod P$. This implies that one can obtain the isotropic generating function from the anisotropic one with the mappings $x \mapsto t^{1/P}$ and $y \mapsto t^{1/P}$.

This paper considers the calculation of the generating functions of isotropic and anisotropic cases of the n-friendly-walk models, for all n, and the ∞-friendly-walk model on the directed square lattice. We concentrate on the cases $n = 0, 1, 2$ and ∞-friendly. We also later summarise the known results from higher dimensional analogues on directed d-dimensional hyper-cubic lattices. Firstly we briefly mention previous work in this area.

In [4] recurrence relations, and the corresponding differential equations, for certain stars and watermelons, both on the directed square lattice and on the general d-dimensional directed hyper-cubic lattice, were obtained. In two dimensions, the Gessel-Viennot determinant for watermelons was evaluated by standard techniques, while in the case of stars the results obtained were conjectural. More recently Guttmann *et al.* [13] showed that the conjectured result for stars follows from the $q \to 1$ limit of the Bender-Knuth theorem, the first published proof of which appears in [8]. Further, the connections between the celebrated 6-vertex model of statistical mechanics, and these vicious walker problems were discussed. Connections between the vicious walker problem and Young tableaux and integer partitions were also presented. In a subsequent paper [12] analogous results for stars and watermelons adjacent to an impenetrable wall were developed.

We begin our investigation in the following section by dealing with stars and watermelons with two walks ($P = 2$) in the n-friendly-walk models, for all $n \in [0, 1, 2, 3, \dots)$ and in the ∞-friendly-walk model. We solve exactly in every case for the complete anisotropic generating functions by decomposing the graphs involved into parts whose generating functions are known.

In section 3 we summarise briefly the computational method, based upon the series analysis technique of differential approximants, that we utilise in later sections to conjecture exact solutions and the location and nature of the singularities of the generating functions of interest.

Section 4 deals with three walkers, firstly with the isotropic cases and then with the anisotropic cases of our models. In both cases we analyse stars and watermelons in the vicious, osculating and 2-friendly-walk models. For the isotropic model we add to the known results for vicious walkers an analytical generating function for osculating stars and a differential equation for osculating watermelons. Further we use the techniques described in section 3 to guess the exponents for the 2-friendly-walk model. In the second part, where we deal with the anisotropic model, we were not able to find a closed form solution for the three walker models. Therefore we look at generating functions of subsets of the configurations which when summed give the anisotropic generating function. We then try to find patterns in the generating functions of the subclasses so as to perhaps conjecture the exact generating function. We explain how far we have progressed in this endeavour.

In section 5 we give the dominant exponents for models with more than three walkers obtained from numerical analysis. For four osculating walkers we were able to identify a singularity other than the dominant one and we conjecture that any exact solution would be constrained to contain both these singularities.

Section 6 builds on the work in [4] and concludes that all n-friendly-walk models in three or more dimensions behave essentially like free walkers, and hence we deduce their exponents.

In section 7 we introduce the concept of inversion relations, which are equations that are satisfied by the generating function, since they may provide a route to an exact solution for some of these problems. We were able to find inversion relations that hold for vicious walker watermelons with an arbitrary number of walkers even though we do not have the explicit anisotropic generating functions for three or more walkers.

2 Two walkers

In this section we consider the n-friendly-walk models with two walkers and solve for the generating functions for stars and watermelons for all n exactly. We first derive the generating functions of watermelons by decomposing the graphs into sections whose generating functions are known. By distinguishing between NE and SE steps, we give the 2-variable or *anisotropic* generating functions for the four cases of special interest explicitly. Secondly, we extend the derivation to stars and give their respective anisotropic generating functions. The isotropic generation function can be obtained from the anisotropic generating function using the mappings $x \mapsto \sqrt{t}$ and $y \mapsto \sqrt{t}$.

For the derivation of the two walker watermelon generating functions we cut the graph at all points l that satisfy $z_{2,l} - z_{1,l} = 2$. That is, at all points where the y-coordinates are a distance two apart. The resulting graphs are called *irreducible watermelons*. We note in passing that this concept generalises to any number of walkers. That is to say, irreducible watermelons are those graphs that start and end with the watermelon configuration, i.e. the walkers are a distance two apart at their starting and ending points, but nowhere else.

The different types of irreducible watermelons that can occur with two walkers can be distinguished by the first step taken. The three different configurations and their anisotropic generating functions are:

(1) Both walkers simultaneously take a step in the same direction, either to the NE or to the SE, shown in figure 4(a). The resulting irreducible watermelon is of length one. The generating function is therefore

$$W_{2,n}^{(1)}(x,y) = x^2 + y^2, \tag{2.5}$$

which is independent of n.

(2) The lower walker takes a NE step and the upper walker takes a SE step. They touch and can stay together for up to n sites (for the n-friendly-walk model) and then diverge. As an example, in figure 4(b) a configuration where two n-friendly walkers with $n \geq 4$ stay together for four consecutive sites is shown.

This configuration consists of a single walk of length $\leq n-1$ but with the weight of each step doubled. The converging and diverging steps are added at the beginning and the end. Hence we use the single walker generating function for walks of length $\leq n-1$. The generating function is

$$\begin{aligned} W^{(2)}_{2,n}(x,y) &= x^2y^2 \sum_{i=0}^{n-1}\sum_{j=0}^{i} \binom{i}{j} x^{2j} y^{2(i-j)} \\ &= x^2\,y^2 \frac{1-(x^2+y^2)^n}{1-x^2-y^2} \\ &= x^2\,y^2\,\beta_n(x,y), \end{aligned} \tag{2.6}$$

which defines β_n. For the cases of special interest, vicious, osculating, 2-friendly and ∞-friendly-walks, $W^{(2)}_{2,n}$ becomes

$$W^{(2)}_{2,0}(x,y) = 0, \tag{2.7}$$

$$W^{(2)}_{2,1}(x,y) = x^2y^2, \tag{2.8}$$

$$W^{(2)}_{2,2}(x,y) = x^2y^2(1+x^2+y^2), \tag{2.9}$$

$$W^{(2)}_{2,\infty}(x,y) = \frac{x^2y^2}{1-x^2-y^2}. \tag{2.10}$$

(3) When the lower walker first takes a SE step and the upper walker a NE step the resulting watermelon is essentially a staircase polygon. An example is shown in part (c) of figure 4. By translating the upper walker by two units in the negative y-direction one obtains a staircase polygon on the sub-lattice defined by $x+y=0 \pmod 2$. The generating function can be found, for example in [2], and is

$$W^{(3)}_{2,n}(x,y) = \frac{1}{2}\left(1-x^2-y^2-\sqrt{\Delta}\right), \tag{2.11}$$

with $\Delta(x,y) = 1-2\,x^2+x^4-2\,y^2-2\,x^2\,y^2+y^4$. Again, the result is model independent.

Adding the generating functions for the three different groups gives the generating function for irreducible watermelons. Since a watermelon consists of a unique sequence of irreducible watermelons and any sequence of irreducible watermelons forms a watermelon, the generating function for watermelons is

$$W_{2,n}(x,y) = \frac{1}{1-\sum_{i=1}^{3} W^{(i)}_{2,n}(x,y)}. \tag{2.12}$$

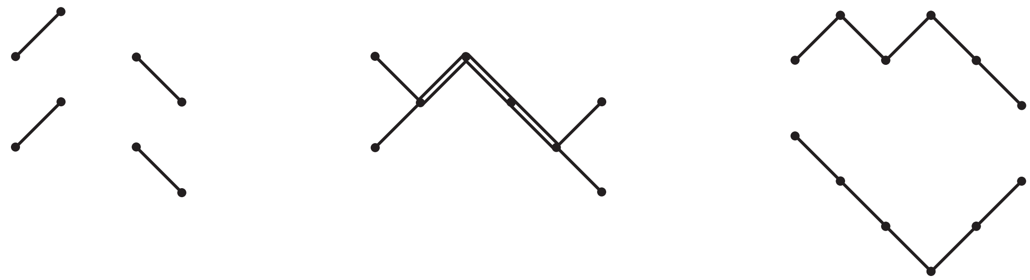
Evaluating this equation for the four special cases of particular interest gives:

$$W_{2,0}(x,y) = \frac{2}{1-x^2-y^2+\sqrt{\Delta}}, \tag{2.13}$$

$$W_{2,1}(x,y) = \frac{2}{1-x^2-y^2-2\,x^2\,y^2+\sqrt{\Delta}}, \tag{2.14}$$

$$W_{2,2}(x,y) = \frac{2}{1-x^2-y^2-2\,x^2\,y^2-2\,x^4\,y^2-2\,x^2\,y^4+\sqrt{\Delta}}, \tag{2.15}$$

$$W_{2,\infty}(x,y) = \frac{2\,(1-x^2-y^2)}{1-2\,x^2-2\,y^2+x^4+y^4+(1-x^2-y^2)\sqrt{\Delta}}. \tag{2.16}$$



(a) (b) (c)

Figure 4: Part (a) shows the two graphs described by $W^{(1)}_{2,n}(x,y)$, Part (b) shows two walkers sharing 4 consecutive sites, with is allowed in the n-friendly-walk model with $n \geq 4$ and Part (c) shows an example of a graph described by $W^{(3)}_{2,n}(x,y)$

We can simplify this by setting $\alpha(x,y) = 1 - x^2 - y^2$. Then the generating functions for all n can be written

$$W_{2,n}(x,y) = \frac{2}{\alpha - 2\,x^2\,y^2\,\beta_n + \sqrt{\Delta}}, \tag{2.17}$$

with β_n defined above. In the four cases of special interest we have:

$$\beta_0(x,y) = 0, \tag{2.18}$$

$$\beta_1(x,y) = 1, \tag{2.19}$$

$$\beta_2(x,y) = 1 + x^2 + y^2 \quad \text{and} \tag{2.20}$$

$$\beta_\infty(x,y) = 1/\alpha(x,y). \tag{2.21}$$

The generating function for *stars,* $S_{2,n}(x,y)$ can be derived by cutting the graph into two parts at the point where the walkers are a distance two apart *for the last time* when traversing the star from beginning to end. That is at $l = \max\{i|z_{1,i} = z_{2,i} - 2\}$. The graph on the left of the cut is the maximal watermelon which the star contains. If the part of the graph to the right is not empty, the first step the lower walker takes is SE and the upper walker must take a NE step. After that the two walkers stay at least four units apart. Now one can translate that portion of the upper walker from $k = l + 1$ by two units in the negative y-direction. The result is a *vicious* walker star configuration. Thus we see that

$$S_{2,n}(x,y) = W_{2,n}(x,y)(1 + xyS_{2,0}(x,y)). \tag{2.22}$$

Applying this result to vicious walker stars gives the equation

$$\begin{aligned} S_{2,0}(x,y) &= \frac{W_{2,0}(x,y)}{1 - x\,y\,W_{2,0}(x,y)} \\ &= \frac{2}{1 - x^2 - y^2 - 2\,x\,y + \sqrt{\Delta}}. \end{aligned} \tag{2.23}$$

Hence we obtain:

$$S_{2,1}(x,y) = 2 \,/\, \left[\sqrt{\Delta}\; (1 - x\,y) - (x+y-1)\; (x+y+1) \right.$$
$$\left. (1 + x\,y)\right], \tag{2.24}$$
$$S_{2,2}(x,y) = 2 \,/\, \left[\sqrt{\Delta}\; \left(1 - x\,y\; \left(1 + x^2 + y^2\right)\right) - (x+y-1) \right.$$
$$\left. (x+y+1)\; \left(1 + x\,y\; \left(1 + x^2 + y^2\right)\right)\right], \tag{2.25}$$
$$S_{2,\infty}(x,y) = 2\; \left(x^2 + y^2 - 1\right) \,/\, \left[\,(x+y-1)\; (x+y+1) \right. \tag{2.26}$$
$$\left. \left(1 - x^2 + x\,y - y^2\right) + \sqrt{\Delta}\; \left(-1 + x^2 + x\,y + y^2\right)\right].$$

As for watermelons, we can rewrite this in a more compact form. Let $\Lambda(x,y) = (x+y-1)(x+y+1)$. Then

$$S_{2,n}(x,y) = \frac{2}{\sqrt{\Delta}(1 - x\,y\,\beta_n) - \Lambda(1 + x\,y\,\beta_n)} \tag{2.27}$$

for $n = 0, 1, 2, \infty$.

For the calculation of those cases with more than two walkers we need to resort to numerical analysis. We introduce a technique based on differential approximants briefly in the following section before we return to the analysis of the walker problem, first to three walkers and then the case when there are more than three walkers in a later section.

3 Analysis of numerical data

In situations where we cannot derive exact results, we have written computer programs to generate the first N terms of the star and watermelon generating functions, typically $N \approx 100$. We use the techniques of series analysis, reviewed in [9], to obtain estimates of the location of singularities, and their associated exponents and in favourable circumstances to conjecture a recurrence relation. We base our numerical analysis on the method of differential approximants which we will briefly discuss below.

The method of differential approximants consists of representing the generating function under study by the integral of the Kth-order differential equation

$$\sum_{i=0}^{K} Q_i(z) D^i f(z) = P(z) \tag{3.28}$$

with $D = z\, d/dz$ and Q_i and P polynomials in z with the respective degrees N_i and L, so that

$$Q_i(z) = \sum_{j=0}^{N_i} Q_{i,j} z^j \qquad \text{and} \qquad P(z) = \sum_{j=0}^{L} P_j z^j. \tag{3.29}$$

With the choice of $Q_{K,0} = 1$, and in the homogeneous case when $P(z) = 0$, $Q_{0,0} = 0$, we force the origin to be a regular singular point. This choice is not necessary but it is motivated by a number of exact solutions such as Onsager's solution of the zero-field free energy of the square lattice Ising model [18].

The differential equation (3.28) is a natural generalisation of the logarithmic derivative Padé representation, which corresponds to the case $K = 1$, $P(z) = 0$.

If $K = 1$ in (3.28) the singularities of the functions that satisfy the differential equation have the form

$$f(z) \sim A(z) + B(z)(1 - z/z_i)^{-\gamma} \tag{3.30}$$

with A and B regular in the neighbourhood of $z = z_i$. In contrast the representable singularities in the $D\log$ Padé method have the form

$$f(z) \sim B(z)(1 - z/z_i)^{-\gamma}. \tag{3.31}$$

For $\gamma \gtrsim 1$ $D\log$ Padé approximants will usually give adequate results, but for smaller γ, and in particular for $\gamma < 0$, the additive term $A(z)$ is often not negligible.

To find sub-dominant singularities one can generalise (3.30) so that one can represent singularities of the following form:

$$\begin{aligned} f(z) \sim A(z)+&B(z)(1 - z/z_c)^{-\gamma}(1 + C_1(z)(1 - z/z_c)^{\Delta_1} + \\ &C_2(z)(1 - z/z_c)^{\Delta_2} + \ldots + C_{K-1}(z)(1 - z/z_c)^{\Delta_{K-1}}) \end{aligned} \tag{3.32}$$

where $\Delta_i > 0$, $\Delta_i \notin \mathbb{Z}$ and $C_i(z)$ is regular in the neighbourhood of z_c. This method can not only be used to accurately estimate the location of the dominant singularity and its exponent, but can also reveal a more complex singularity structure with sub-dominant singularities, as well as estimates of the associated exponents. Which singularities one can identify depends on the distribution and dominance of the singularities, and on the length of the available series. Singularities which are close to the dominant singularity or are beyond, on the ray from the origin containing the physical singularity, are difficult or impossible to find.

Our analysis was carried out using the computer program newgrqd [9] which implements the method. A detailed discussion of the method is given in [9]. We used this method to estimate the location of singularities and the corresponding exponents for those isotropic models for which we could not find a generating function or a differential equation. This includes the 2- and ∞-friendly-walk models with three walkers, and the osculating walker and 2-friendly-walk model with four walkers, all for both watermelons and stars. For these problems, inhomogeneous first ($K = 1$) and second ($K = 2$) order differential approximants appeared to give completely adequate results. If the numerical estimate for a singularity in one of these models is something like 0.124998 we then conjecture the true value is 1/8 since similar models which have been solved have singularities whose real and imaginary parts are relative simple fractions. Similarly, we can conjecture exact values for the associated exponents, which are also simple fractions or integers in all known cases.

Apart from the purely numerical estimates for the singularities which one can find with this method, one can in favourable circumstances also conjecture an exact solution, as follows: We regard f as formal power series and require term by term agreement between the given series and the expansion of (3.28). This yields a system of linear equations. The number of equations one has depends on the length of the series, the order of the differential equation and the degree of the polynomials Q_i and P. Solving this system for the $Q_{i,j}$ gives a recurrence relation. This recurrence than implicitly, yields all terms in the expansion. In particular, it predicts terms in the expansion of f beyond those used to find the recurrence relation. To be confident in conjecturing that the recurrence is exact, one needs many more coefficients in the series to be known than are used in finding the the conjectured recurrence. But if the predicted coefficients and the known coefficients are in agreement, one can confidently conjecture the exact solution. To fully exploit the calculated series, one has to vary both the degree and order of the recurrence. We also used the Maple package "gfun" [22] and a Mathematica routine called "Solver" [19] to find conjectured recurrences.

4 Three walkers

In this section we deal with stars and watermelons with three walkers only. Of the n-friendly-walk models we consider the vicious walker ($n = 0$), osculating walker ($n = 1$), 2-friendly-walk case both isotropically and anisotropically and further the isotropic ∞-friendly-walk model.

First we extend isotropic results given for vicious walkers in [4] to osculating walkers and analyse the isotropic 2-friendly-walk and ∞-friendly-walk series. In the second part we analyse the anisotropic series (in which we retain the distinction between NE and SE steps) for the vicious, osculating walker and 2-friendly walk models and compare them.

4.1 The isotropic results

In this section we find the singularities of the generating functions for stars and watermelons with three walkers in the n-friendly-walk model with $n = 0, 1, 2$ and the ∞-friendly-walk model. Further we give the generating function for osculating ($n = 1$) stars and a differential equation for osculating ($n = 1$) watermelons.

For the models mentioned above with the exception of vicious walkers, which have been solved before by Essam and Guttmann [4], we generated the first few terms in the series expansion. We first tried to find a recurrence based on the method explained above to find an exact solution. If we were able to find a recurrence, we then tried to solve the recurrence. If we were not able to find a recurrence, we analysed the series with the numerical method of differential approximants in order to estimate the singularities.

4.1.1 Vicious walkers

In order to compare the vicious walker results with the ones for the other models we quote the vicious walker results Essam and Guttmann obtained in [4]. The isotropic vicious walker generating function is given in [4] (eqn. (74)) as

$$S_{3,0}(t) = [1 - 4t - \sqrt{1-8t}]/(8t^2). \tag{4.33}$$

It clearly has a singularity at $t = 1/8$ with exponent $1/2$.

Further Essam and Guttmann [4] proved that vicious walker watermelons satisfy the differential equation:

$$\begin{aligned} t^2(1+t)(1-8t)W''_{3,0}(t) + t(8 - 42t - 32t^2)W'_{3,0}(t) + \\ (12 - 40t - 16t^2)W_{3,0}(t) = 12, \end{aligned} \tag{4.34}$$

given as eqn. (63) in [4]. This differential equation has the solution ([4] eqn. (65))

$$W_{3,0}(t) = [-1 + t - 3t^2 + F(-\frac{1}{8}, \frac{1}{4}; -1, 2, 2, -2; t)]/(8t^2), \tag{4.35}$$

where $F(a, b; c, d, e, f; t)$ is a *Heun function* [20]. The vicious walker watermelon generating function has the a physical singularity at $t = t_c = 1/8$ (at the same location as the star generating function) and an additional non-physical singularity at $t = t_c = -1$. Both singularities have exponent 3, corresponding to a singularity of the form $(1 - t/t_c)^3 \ln(1 - t/t_c)$.

4.1.2 Osculating walkers

Using the method described in section 3 we found a conjectured recurrence for three osculating stars which has the solution

$$S_{3,1}(t) = \frac{-4t^2 - 15t + 3 - 3(1-t)\sqrt{1-8t}}{8t^2(t+1)}. \tag{4.36}$$

The osculating stars therefore have singularities at $t = 1/8$ with exponent $1/2$ and at $t = -1$ with exponent -1. The three walker osculating watermelons generating function $W_{3,1}(t) = \sum_{i=0}^{\infty} u(i)\, t^i$ has coefficients which were found to satisfy the following recurrence:

$$\begin{aligned} &(-45/4 - 19i/8 - i^2/8)\, u(i+5) + (51 + 29i/2 + i^2)\, u(i+4) + \\ &(40 + 29i/4 + i^2/4)\, u(i+3) + (-32 - 35i/2 - 2i^2)\, u(i+2) + \\ &(-31/4 - 39i/8 - i^2/8)\, u(i+1) + (2 + 3i + i^2)\, u(i) = 0. \end{aligned} \tag{4.37}$$

This was found by the technique described above. The corresponding differential equation satisfied by the generating function is:

$$\begin{aligned} &t^2\,(-1+t)^2\,(1+t)^2\,\,(-1+8\,t)\,W''_{3,1}(t) + \\ &2\,t\,\,(-1+t)\,\,(1+t)^2\,\left(5 - 35t + 16t^2\right)W'_{3,1}(t) + \\ &4\,\left(-5 + 18\,t + 41\,t^2 - 10\,t^3 - 6\,t^4 + 4\,t^5\right)W_{3,1}(t) = 4(-1+t)(5+2t). \end{aligned} \tag{4.38}$$

Analysis of this differential equation shows that the osculating watermelon generating function has a singularity at $t = 1/8$ with exponent 3 (which has a confluent logarithmic term), a double singularity at $t = -1$ with exponents -1 and 2, where the latter also has a confluent logarithmic term, and a singularity at $t = 1$ with exponent -1, corresponding to a simple pole.

Comparing the differential equations (4.34) and (4.38), one sees in the osculating walker case that the second derivative term has an extra factor of $(1+t)(-1+t)^2$, the first derivative term has an extra factor of $(-1+t)(1+t)^2$ and the polynomial multiplying the function itself is quintic rather than quadratic compared to the vicious walker case. We are therefore unable to solve this differential equation.

Although there is as yet no proof for the conjectured osculating walker recurrences for stars and watermelons that we have given, the number of terms in the series that are satisfied by the equations is so large that it is inconceivable that the recurrence does not hold for all terms in the series. We have twice as many coefficients as are needed to conjecture (4.36) and (4.38).

4.1.3 2- and ∞-friendly walkers

Unlike for the problem of three osculating walkers, we could not find an exact recurrence for stars or watermelons with three walkers in the 2-friendly-walks or ∞-friendly-walks model. Thus for these models we used the capability of the differential approximant technique to *conjecture* the singularities, as implemented in the program newgrqd [9], and as discussed at the end of section 3.

For 2-friendly-walk stars the differential approximant technique identifies singularities at $t = 1/8$ and $t = -1/4 \pm 1/2i$ with the respective exponents $1/2$ and $-1/2$. Watermelons have singularities at the same location with exponents 3 and $1/2$, respectively. In the former case we expect a confluent logarithmic term. Using the same method we also identified singularities and exponents for ∞-friendly-walk stars and watermelons. The singularities for ∞-friendly stars are at $t = 1/8$ and at $t = -1$ with the respective exponents $1/2$ and 3. For ∞-friendly-walks watermelons we found the same singularities which both have the exponent 3. Again, we expect a confluent logarithmic term.

We observe that the physical singularity for stars is at $t = 1/8$ for all models considered in this section, with exponent $1/2$, corresponding to a square root branch point. For watermelons the primary singularity is at the same position as for stars, but the exponent there is 3, corresponding to a singularity of form $(1-8t)^3 \ln(1-8t)^\zeta$, where ζ is unknown. For further discussion see section 5. The singularities of the different models are summarised in Table 1.

4.2 The anisotropic models

4.2.1 Partial generating functions

To further our analysis we anisotropised the model, by which we mean that we distinguish between NE and SE steps. This highlights similarities and differences of the models which are hidden in the isotropic case. Furthermore, the anisotropic series analysis gives indications of type of the isotropic solution one might expect, [3, 10]. By appropriately re-summing the series we conclude that one should expect D-finite solutions for all models considered. In this subsection we explain how we rewrite the anisotropic generating function as an infinite sum of generating functions for subsets of the original set of configurations. We then apply this procedure to both watermelons and stars in the vicious, osculating and 2-friendly-walk models. The generating functions for the subsets are all rational functions, and we tried to discover patterns in those generating functions. For each model we provide a complete discussion of our progress towards an exact solution. In the case of three vicious walker watermelons we have found an inversion relation, which is described in section 7.

To define the partial generating functions let us consider the star generating function:

$$S_{3,n}(x,y) = \sum_{j,k\geq 0} s_{3,n,j,k}\, x^j y^k, \tag{4.39}$$

where, as usual, the subscript n identifies the model and $s_{3,n,j,k}$ is the number of three walker stars with j NE and k SE steps in the model indexed by n. Since in all stars the total number of NE and SE steps must be divisible by three, all $s_{3,n,j,k}$ are zero for $j+k \neq 0 \pmod 3$. We calculated the terms up to $j+k \leq 210$. By partially summing this equation we obtained

$$S_{3,n}(x,y) = \sum_{k\geq 0} H^{(s)}_{n,k}(x)\, y^k, \tag{4.40}$$

where the superscript s denotes stars, and w will denote watermelons. The generating functions $H^{(s)}_{n,k}(x)$ are defined by

$$H^{(s)}_{n,k}(x) = \sum_{j\geq 0} s_{3,n,j,k}\, x^j \tag{4.41}$$

where x is the variable associated with the total number of NE steps in the configurations. The $H^{(s)}_{n,k}$ are the generating functions for configurations with exactly k SE steps. Since $s_{3,n,j,k} = s_{3,n,k,j}$, $H^{(s)}_{n,k}$ generates configurations with exactly k NE steps too.

The same approach can be applied to the series for watermelons with the further constraint that the exponents of x and y must both be divisible by three. Since the walkers start and end in the same pattern, all three walks contain the same number of NE steps and the same number of SE steps. Thus the constraint follows.

For all models with three walkers the $H^{(s)}_{n,k}$ were found to be rational functions with denominator $(1-x^3)^{k+1}$. Since the exponents of y must be divisible by three for watermelons, all $H^{(w)}_{n,k} = 0$ for $k \neq 0 \pmod 3$. The x^3 in the factor $1-x^3$ in the denominator arises because the graphs grow by three steps when the length is increased by one. The exponent $k+1$ can be understood by considering how the stars grow. A factor $(1-x^3)$ arises from the extra step, as this adds a further three NE steps, and similarly a factor $(1-x^3)^k$ arises from the k SE steps in the walks. In general the denominator has the form $(1-x^P)^{k+1}$ for a P walker configuration which grows in one direction and into which k steps in the other direction are inserted.

For the first few k, such as $k = 0, 1, 2, 3$, one can find the $H^{(s)}_{n,k}$ and $H^{(w)}_{n,k}$ explicitly by considering how one can insert k SE steps into three walkers going NE-wards without violating the rules of the n-friendly-walk model. For the next few k one can try the find a rational function fitting the series. For this one needs relatively long series. As a computational aid, one can limit the degree of the numerator and denominator polynomials appropriately, using the previously calculated $H_{n,k}$ as a guide.

Once the denominator pattern is established and the approximate growth of the degree of the numerator polynomial is known, one can easily fit the coefficients of the numerator to the series, keeping the system of equations always over-determined by keeping the series long enough.

For the models discussed in this section, that are n-friendly-walk stars and watermelons with $n = 0, 1, 2$, the generating functions for configurations with a fixed number of SE steps only have singularities at $x^3 = 1$. The simple denominator pattern of all the different generating functions $H_k(x)$ suggests D-finiteness and solvability [10] for all models considered.

It is worth remarking that, as we will, see the anisotropic series of the partial generating functions for stars are more complex than those for watermelons, while the isotropic solutions for stars are simpler than those for watermelons.

We now calculate and discuss the numerator polynomials for the vicious walker, osculating walker and 2-friendly-walk models.

4.2.2 Vicious walkers

First we use the Gessel-Viennot formula for vicious walker configurations to enumerate the first terms in the series for watermelons and stars. Partial summation gives the generating functions for watermelons

$$H^{(w)}_{0,0}(x) = 1/\left(1-x^3\right), \tag{4.42}$$

$$H^{(w)}_{0,3}(x) = 1/\left(1-x^3\right)^4, \tag{4.43}$$

$$H^{(w)}_{0,6}(x) = \left(1+3\,x^3+x^6\right)/\left(1-x^3\right)^7, \tag{4.44}$$

$$H^{(w)}_{0,9}(x) = \left(1+10\,x^3+20\,x^6+10\,x^9+x^{12}\right)/\left(1-x^3\right)^{10}, \tag{4.45}$$

$$H^{(w)}_{0,12}(x) = \left(1+22\,x^3+113\,x^6+190\,x^9+113\,x^{12}+22\,x^{15}+x^{18}\right) \\ /\left(1-x^3\right)^{13}. \tag{4.46}$$

We first note that the numerator polynomials are symmetric for vicious watermelons, which we shall see later is in contrast to the other models we consider. We show below, in section 7, that this gives rise to a so-called *inversion relation*. Further, the number of coefficients in the numerator increases only by two from one partial generating function to the next. This is noteworthy since it allows us to calculate *all* coefficients iteratively by using the x–y symmetry in the anisotropic generating function and the symmetry of the numerator polynomials of the partial generating functions. This we demonstrate explicitly in section 7.

In an analogous fashion one obtains the generating functions for stars with fixed number of

SE steps

$$H^{(s)}_{0,0}(x) = 1/\left(1-x^3\right), \tag{4.47}$$
$$H^{(s)}_{0,1}(x) = x^2/\left(1-x^3\right)^2, \tag{4.48}$$
$$H^{(s)}_{0,2}(x) = \left(x+x^4\right)/\left(1-x^3\right)^3, \tag{4.49}$$
$$H^{(s)}_{0,3}(x) = \left(1+2\,x^3+x^6\right)/\left(1-x^3\right)^4, \tag{4.50}$$
$$H^{(s)}_{0,4}(x) = \left(4\,x^2+4\,x^5+x^8\right)/\left(1-x^3\right)^5, \tag{4.51}$$
$$H^{(s)}_{0,5}(x) = \left(2\,x+12\,x^4+6\,x^7+x^{10}\right)/\left(1-x^3\right)^6, \tag{4.52}$$
$$H^{(s)}_{0,6}(x) = \left(1+12\,x^3+28\,x^6+9\,x^9+x^{12}\right)/\left(1-x^3\right)^7, \tag{4.53}$$
$$H^{(s)}_{0,7}(x) = \left(9\,x^2+48\,x^5+57\,x^8+12\,x^{11}+x^{14}\right)/\left(1-x^3\right)^8, \tag{4.54}$$
$$H^{(s)}_{0,8}(x) = \left(3\,x+54\,x^4+144\,x^7+105\,x^{10}+16\,x^{13}+x^{16}\right) / \left(1-x^3\right)^9, \tag{4.55}$$
$$H^{(s)}_{0,9}(x) = \left(1+34\,x^3+230\,x^6+370\,x^9+179\,x^{12}+20\,x^{15}+x^{18}\right) / \left(1-x^3\right)^{10}, \tag{4.56}$$
$$H^{(s)}_{0,10}(x) = \left(16\,x^2+225\,x^5+795\,x^8+837\,x^{11}+289\,x^{14}+ 25\,x^{17}+x^{20}\right)/\left(1-x^3\right)^{11}. \tag{4.57}$$

Note that the numerators, while apparently unimodal polynomials, are not symmetric.

4.2.3 Osculating walkers

We now repeat the above construction for osculating walkers. For both osculating walkers and the 2-friendly-walk model we enumerated configurations using computer programs we wrote which are based on the method of transfer matrices. This is a standard method in statistical mechanics [1, 17, 18] and combinatorics [21] to calculate terms in the series expansion of a partition function of a lattice model.

We then partially summed the anisotropic series to obtain the generating functions for configurations with fixed numbers of SE steps. We found the first few $H^{(w)}_{1,k}(x)$ to be

$$H^{(w)}_{1,0}(x) = 1/(1-x^3), \tag{4.58}$$

$$H^{(w)}_{1,3}(x) = (1+4x^3-5x^6+x^{12})/(1-x^3)^4, \tag{4.59}$$

$$H^{(w)}_{1,6}(x) = (1+14x^3+8x^6-46x^9+32x^{12}+4x^{15}- 11x^{18}+3x^{21})/(1-x^3)^7, \tag{4.60}$$

$$H^{(w)}_{1,9}(x) = (1+30\,x^3+131\,x^6-100\,x^9-255\,x^{12}+422\,x^{15}- 187\,x^{18}-70\,x^{21}+106\,x^{24}-42\,x^{27}+6\,x^{30}) /(1-x^3)^{10}, \tag{4.61}$$

$$H^{(w)}_{1,12}(x) = (1+53\,x^3+540\,x^6+967\,x^9-1826\,x^{12}-551\,x^{15}+ 3460\,x^{18}-3293\,x^{21}+869\,x^{24}+858\,x^{27}-962\,x^{30}+ 438\,x^{33}-102\,x^{36}+10\,x^{39})/(1-x^3)^{13}. \tag{4.62}$$

The numerator polynomials are no longer symmetric, and their degree is $3k+3$. Thus the number of coefficients increases by three from $H^{(w)}_{1,k}(x)$ to $H^{(w)}_{1,k+3}(x)$, $(k>0)$. From the available data we could determine the leading coefficients in the numerator of the $H^{(w)}_{1,k}(x)$. They are generated by the following polynomials where $A^{(w)}_{1,i}(k)$ is the coefficient of x^{d-i} in the numerator of $H^{(w)}_{1,k}(x)$ and d is the degree of the numerator of $H^{(w)}_{1,k}(x)$:

$$A^{(w)}_{1,0}(k) = k/6+k^2/18 \qquad (k\geq 3), \tag{4.63}$$

$$A^{(w)}_{1,3}(k) = 5k/6-k^2/9-k^3/18 \qquad (k\geq 6), \tag{4.64}$$

$$A^{(w)}_{1,6}(k) = 4-k/6-35k^2/36+k^4/36 \qquad (k\geq 6), \tag{4.65}$$

$$A^{(w)}_{1,9}(k) = -4-29k/6-13k^2/36+55k^3/108+k^4/36-k^5/108 \qquad (k\geq 9), \tag{4.66}$$

$$A^{(w)}_{1,12}(k) = 2k+187k^2/72+199k^3/432-67k^4/432-7k^5/432+ k^6/432 \qquad (k\geq 12), \tag{4.67}$$

$$A^{(w)}_{1,15}(k) = 2k/3+83k^2/360-1499k^3/2160-17k^4/72+k^5/36+ k^6/180-k^7/2160 \qquad (k\geq 12). \tag{4.68}$$

Correspondingly one can find polynomials that generate the low order coefficients in the numerators of $H^{(w)}_{1,k}(x)$. Let $B^{(w)}_{1,i}(k)$ be the coefficient of x^i in $H^{(w)}_{1,k}(x)$. Then we find the first of these generating polynomials to be:

$$B_{1,0}^{(w)}(k) = 1, \tag{4.69}$$

$$B_{1,3}^{(w)}(k) = -1 + 17k/18 + 2k^2/9 + k^3/162, \tag{4.70}$$

$$B_{1,6}^{(w)}(k) = 4 - 13k/6 - 385k^2/648 + 7k^3/144 + 193k^4/11664 + k^5/1296 + k^6/10497, \tag{4.71}$$

$$B_{1,9}^{(w)}(k) = -9 + 403k/180 + 1523k^2/1296 - 6823k^3/58320 - 2821k^4/46656 - 607k^5/233280 + 91k^6/209952 + 107k^7/3149280 + k^8/1259712 + k^9/170061120, \tag{4.72}$$

$$B_{1,12}^{(w)}(k) = 16 - 473k/180 - 47k^2/24 + 50419k^3/1166400 + 2259829k^4/20995200 + 73121k^5/8398080 - 227681k^6/151165440 - 23729k^7/125971200 + 1061k^8/2267481600 + 427k^9/680244480 + 301k^{10}/12244400640 + 11k^{11}/30611001600 + k^{12}/550998028800, \tag{4.73}$$

$$B_{1,15}^{(w)}(k) = -25 + 718k/315 + 716203k^2/226800 + 3808061k^3/40824000 - 4870531k^4/36741600 - 2387911k^5/132269760 + 184949k^6/105815808 + 238680983k^7/476171136000 + 13861k^8/1763596800 - 41381k^9/12244400640 - 66617k^{10}/428554022400 + 15779k^{11}/4821232752000 + 1349k^{12}/3856986201600 + 59k^{13}/6942575162880 + k^{14}/11570958604800 + k^{15}/3124158823296000. \tag{4.74}$$

$B_{1,0}^{(w)}(k)$ is valid for all k. All other given $B_{1,i}^{(w)}(k)$ are valid for $k \geq 3$.

The first partial generating functions for osculating stars were found to be

$$H_{1,0}^{(s)}(x) = 1/(1-x^3), \tag{4.75}$$

$$H_{1,1}^{(s)}(x) = (3\,x^2 - 2\,x^5)/(1-x^3)^2, \tag{4.76}$$

$$H_{1,2}^{(s)}(x) = (3\,x - 2\,x^7 + x^{10})/(1-x^3)^3, \tag{4.77}$$

$$H_{1,3}^{(s)}(x) = (1 + 8\,x^3 - 4\,x^6 - 3\,x^9 + 2\,x^{12})/(1-x^3)^4, \tag{4.78}$$

$$H_{1,4}^{(s)}(x) = (9\,x^2 + 13\,x^5 - 24\,x^8 + 12\,x^{11} - x^{17})/(1-x^3)^5, \tag{4.79}$$

$$H_{1,5}^{(s)}(x) = (4\,x + 34\,x^4 - 14\,x^7 - 15\,x^{10} + 18\,x^{13} - 7\,x^{16} + x^{19})/(1-x^3)^6, \tag{4.80}$$

$$H_{1,6}^{(s)}(x) = (1 + 31\,x^3 + 73\,x^6 - 74\,x^9 - 4\,x^{12} + 43\,x^{15} - 21\,x^{18} + x^{21} + x^{24})/(1-x^3)^7, \tag{4.81}$$

$$H_{1,7}^{(s)}(x) = (16\,x^2 + 146\,x^5 + 43\,x^8 - 193\,x^{11} + 170\,x^{14} - 56\,x^{17} - 7\,x^{20} + 10\,x^{23} - 2\,x^{26})/(1-x^3)^8, \tag{4.82}$$

$$H_{1,8}^{(s)}(x) = (5\,x + 131\,x^4 + 383\,x^7 - 217\,x^{10} - 97\,x^{13} + 206\,x^{16} - 112\,x^{19} + 28\,x^{22} - 7\,x^{25} + 4\,x^{28} - x^{31})/(1-x^3)^9, \tag{4.83}$$

$$H_{1,9}^{(s)}(x) = (1 + 71\,x^3 + 624\,x^6 + 655\,x^9 - 894\,x^{12} + 297\,x^{15} + 346\,x^{18} - 410\,x^{21} + 161\,x^{24} - 5\,x^{27} - 14\,x^{30} + 3\,x^{33})/(1-x^3)^{10}, \tag{4.84}$$

$$H_{1,10}^{(s)}(x) = (25\,x^2 + 564\,x^5 + 2096\,x^8 - 6\,x^{11} - 1502\,x^{14} + 1688\,x^{17} - 867\,x^{20} + 160\,x^{23} + 73\,x^{26} - 62\,x^{29} + 25\,x^{32} - 7\,x^{35} + x^{38})/(1-x^3)^{11}. \tag{4.85}$$

The degree of the numerator polynomial increases when going from $H_{1,k}^{(s)}$ to $H_{1,k+1}^{(s)}$ by two for k even and by five for k odd ($k > 0$). The lowest order non-zero coefficients of x^i in the $H_{1,k}^{(s)}$

denoted $B^{(s)}_{1,i}(k)$ are generated by a polynomial of degree i. The first few are found to be:

$$B^{(s)}_{1,0}(k) = 1 \qquad \text{for } k = 0 \pmod 3 \text{ and } k \geq 0 \tag{4.86}$$

$$B^{(s)}_{1,1}(k) = 7/3 + k/3 \qquad \text{for } k = 2 \pmod 3 \text{ and } k \geq 2 \tag{4.87}$$

$$B^{(s)}_{1,2}(k) = 25/9 + 10k/9 + k^2/9 \qquad \text{for } k = 1 \pmod 3 \text{ and } k \geq 4 \tag{4.88}$$

$$B^{(s)}_{1,3}(k) = -2 + 29k/18 + k^2/2 + 2k^3/81 \qquad \text{for } k = 0 \pmod 3 \text{ and } k \geq 3 \tag{4.89}$$

$$B^{(s)}_{1,4}(k) = -451/81 + 13k/108 + 175k^2/216 + 41k^3/324 + k^4/216 \qquad \text{for } k = 2 \pmod 3 \text{ and } k \geq 5 \tag{4.90}$$

$$B^{(s)}_{1,5}(k) = -235/81 - 3184k/1215 + 65k^2/243 + 49k^3/216 + 13k^4/486 + 7k^5/9720 \qquad \text{for } k = 1 \pmod 3 \text{ and } k \geq 4 \tag{4.91}$$

$$B^{(s)}_{1,6}(k) = 10 - 479k/180 - 763k^2/540 + 89k^3/1296 + 223k^4/3888 + 271k^5/58320 + 17k^6/174960 \qquad \text{for } k = 0 \pmod 3 \text{ and } k \geq 6 \tag{4.92}$$

4.2.4 Two-friendly-walk model

The analogous construction for the 2-friendly-walk model leads to the partial generating functions for configurations with exactly k SE steps. The degree of the numerator of $H^{(w)}_{2,k}(x)$ was found to be $5k+3$. The first five generating functions are given below:

$$H^{(w)}_{2,0}(x) = 1/(1-x^3), \tag{4.93}$$

$$H^{(w)}_{2,3}(x) = (1 + 4x^3 + x^6 - 8x^9 + 2x^{12} + x^{18})/(1-x^3)^4, \tag{4.94}$$

$$H^{(w)}_{2,6}(x) = (1 + 20x^3 + 10x^6 + 2x^9 - 86x^{12} + 31x^{15} + 55x^{18} - 21x^{21} - 2x^{24} - 10x^{27} + 4x^{30} + x^{33})/(1-x^3)^7, \tag{4.95}$$

$$H^{(w)}_{2,9}(x) = \big(1 + 46\,x^3 + 201\,x^6 + 48\,x^9 - 357\,x^{12} - 330\,x^{15} + 38\,x^{18} + 1184\,x^{21} - 736\,x^{24} - 286\,x^{27} + 169\,x^{30} + 52\,x^{33} + 81\,x^{36} - 72\,x^{39} - 4\,x^{42} + 6\,x^{45} + x^{48}\big) \,/\, \big(1-x^3\big)^{10}, \tag{4.96}$$

$$H^{(w)}_{2,12}(x) = \big(1 + 82\,x^3 + 941\,x^6 + 1863\,x^9 - 112\,x^{12} - 6099\,x^{15} - 509\,x^{18} + 4028\,x^{21} + 7562\,x^{24} - 4024\,x^{27} - 12831\,x^{30} + 11217\,x^{33} - 5\,x^{36} - 906\,x^{39} - 1007\,x^{42} - 485\,x^{45} + 966\,x^{48} - 124\,x^{51} - 106\,x^{54} + x^{57} + 8\,x^{60} + x^{63}\big) / \big(1-x^3\big)^{13}. \tag{4.97}$$

The sequences of the leading five coefficients of the the numerator of $H^{(w)}_{2,k}(x)$ are generated by the functions:

$$A^{(w)}_{2,0}(k) = 1, \tag{4.98}$$
$$A^{(w)}_{2,1}(k) = 2k/3, \tag{4.99}$$
$$A^{(w)}_{2,2}(k) = 5 - 3k + 2k^2/9, \tag{4.100}$$
$$A^{(w)}_{2,3}(k) = -18 + 86k/9 - 2k^2 + 4k^3/81, \tag{4.101}$$
$$A^{(w)}_{2,4}(k) = 72 - 289k/6 + 527k^2/54 - 2k^3/3 + 2k^4/243. \tag{4.102}$$

Again we can find polynomials that generate the coefficients of x^{3i} in the numerator of $H^{(w)}_{2,3i}$. The first three are found to be:

$$B^{(w)}_{2,0}(k) = 1, \tag{4.103}$$
$$B^{(w)}_{2,3}(k) = -12 + 53k/18 + k^2/2 + k^3/162, \tag{4.104}$$
$$B^{(w)}_{2,6}(k) = 243 - 911k/36 - 1337k^2/324 + 367k^3/1296 + 451k^4/11664 + 13k^5/11664 + k^6/104976. \tag{4.105}$$

This construction can be continued, but it requires an ever increasing amount of data and we have not been able to use this information to conjecture a full solution. To find a solution, one would need further rules which determine sufficiently many terms such that one could combine these rules with the x–y symmetry to determine all subsequent terms from a finite set of data.

Looking at the sum of coefficients in the numerator polynomials of $H^{(w)}_{2,k}$ one finds that they are the the 3-dimensional Catalan numbers [24], generated by $2(3i)!/i!(i+1)!(i+2)!$ with $i = 3k$ [24]. Interestingly the numerator evaluated at $x = 1$ is exactly the same for vicious walker, osculating walker and 2-friendly-walk watermelons.

Similarly for stars the following partial generating functions are found

$$H^{(s)}_{2,0}(x) = 1/\left(1-x^3\right), \tag{4.106}$$

$$H^{(s)}_{2,1}(x) = \left(3\,x^2-2\,x^8\right)/\left(1-x^3\right)^2, \tag{4.107}$$

$$H^{(s)}_{2,2}(x) = \left(3\,x+4\,x^4-6\,x^7+x^{16}\right)\left(1-x^3\right)^3, \tag{4.108}$$

$$\begin{aligned} H^{(s)}_{2,3}(x) = {} & \left(1+10\,x^3-x^6-6\,x^9-2\,x^{12}+2\,x^{18}\right) \\ & /\left(1-x^3\right)^4, \end{aligned} \tag{4.109}$$

$$\begin{aligned} H^{(s)}_{2,4}(x) = {} & \left(13\,x^2+11\,x^5+3\,x^8-38\,x^{11}+10\,x^{14}+14\,x^{17}-\right. \\ & \left. 2\,x^{20}-x^{23}-x^{26}\right)/\left(1-x^3\right)^5, \end{aligned} \tag{4.110}$$

$$\begin{aligned} H^{(s)}_{2,5}(x) = {} & \left(6\,x+40\,x^4+25\,x^7-71\,x^{10}-18\,x^{13}+50\,x^{16}-\right. \\ & \left. 6\,x^{19}+2\,x^{22}-10\,x^{25}+2\,x^{28}+x^{31}\right) \\ & /\left(1-x^3\right)^6, \end{aligned} \tag{4.111}$$

$$\begin{aligned} H^{(s)}_{2,6}(x) = {} & \left(1+42\,x^3+100\,x^6-11\,x^9-169\,x^{12}+59\,x^{15}+\right. \\ & 4\,x^{18}+60\,x^{21}-21\,x^{24}-20\,x^{27}+4\,x^{30}+ \\ & \left. x^{33}+x^{36}\right)/\left(1-x^3\right)^7, \end{aligned} \tag{4.112}$$

$$\begin{aligned} H^{(s)}_{2,7}(x) = {} & \left(24\,x^2+199\,x^5+138\,x^8-196\,x^{11}-256\,x^{14}+\right. \\ & 34\,x^{17}+404\,x^{20}-170\,x^{23}-98\,x^{26}+34\,x^{29}+ \\ & \left. 8\,x^{32}+12\,x^{35}-5\,x^{38}-x^{41}\right)/\left(1-x^3\right)^8, \end{aligned} \tag{4.113}$$

$$\begin{aligned} H^{(s)}_{2,8}(x) = {} & \left(7\,x+190\,x^4+569\,x^7+19\,x^{10}-718\,x^{13}-\right. \\ & 463\,x^{16}+1107\,x^{19}-179\,x^{22}-286\,x^{25}+ \\ & 58\,x^{28}-54\,x^{31}+119\,x^{34}-41\,x^{37}-10\,x^{40}+ \\ & \left. 5\,x^{43}\right)/\left(1-x^3\right)^9, \end{aligned} \tag{4.114}$$

$$\begin{aligned} H^{(s)}_{2,9}(x) = {} & \left(1+100\,x^3+894\,x^6+1180\,x^9-547\,x^{12}-\right. \\ & 2338\,x^{15}+885\,x^{18}+1314\,x^{21}-412\,x^{24}+ \\ & 156\,x^{27}-927\,x^{30}+504\,x^{33}+106\,x^{36}-65\,x^{39}- \\ & \left. 8\,x^{42}-16\,x^{45}+7\,x^{48}+x^{51}\right)/\left(1-x^3\right)^{10}, \end{aligned} \tag{4.115}$$

$$\begin{aligned} H^{(s)}_{2,10}(x) = {} & \left(36\,x^2+859\,x^5+3020\,x^8+1823\,x^{11}-4568\,x^{14}-\right. \\ & 2164\,x^{17}+2234\,x^{20}+3590\,x^{23}-855\,x^{26}- \\ & 4436\,x^{29}+2779\,x^{32}+261\,x^{35}-324\,x^{38}- \\ & \left. 19\,x^{41}-150\,x^{44}+111\,x^{47}-x^{50}-8\,x^{53}\right) \\ & /\left(1-x^3\right)^{11}. \end{aligned} \tag{4.116}$$

Again the same construction could be repeated for the 2-friendly stars as for 2-friendly watermelons and osculating stars and watermelons to find polynomials that generate the high order and low order coefficients in the numerator polynomials. Given its futility however we do not do this here. We observe for stars, as we did for watermelons, that for fixed k the numerator of the partial generating functions $H^{(s)}_{n,k}$ evaluated at $x = 1$ is the same for the three different models, vicious walkers, osculating walkers and 2-friendly walkers, ($n = 0, 1, 2$). The sums of the coefficients are the Motzkin numbers [25], which are generated by $(1 - x - (1 - 2x - 3x^2)^{(1/2)})/(2x^2)$ [25].

5 More than three walkers

We have no exact results for more than three walkers, although one could carry out a similar analysis of the anisotropic series generating functions, though this is unlikely to be fruitful. We did however calculate the first terms of anisotropic generating functions of vicious walker watermelons with more than three walkers. We used these to determine an inversion relation which is discussed below in section 7. We further calculated the first terms in the isotropic generating functions for both stars and watermelons in the vicious walker, osculating walker and 2-friendly-walk models with four walkers. We again analysed the series with the differential approximants program newgrqd. For both stars and watermelons all three models have a dominant singularity at $t = 1/16$, where all stars have the exponent 2 (and a confluent logarithmic term) and all watermelons have the exponent 13/2. With the differential approximant technique we could find a sub-dominant singularity for the vicious walker watermelons and for osculating walker stars and watermelons. It has location $t = -1/4$ with the exponent 13/2 for all three models. These results are summarised in table 1. We expect further singularities for the 2-friendly-walk model with four walkers, but we do not have enough data to determine those.

We observe that the dominant singularity for P walker configurations occurs at $t = 2^{-P}$ for all two dimensional models for which we calculated singularities in this paper. Furthermore, the corresponding exponent is found to be the same for models with the same number of walkers and the same type of configuration, stars or watermelons, but with varying n-friendliness (see table 1). They obey the asymptotics which Fisher [5] first derived for the vicious walker model. He considered the chance of a *reunion* of P vicious walkers which corresponds to vicious walker watermelons in our language. The number of configurations we called stars of length L corresponds to the *probability of survival* of P walkers after L steps. The generating functions behave in the vicinity of $t = 2^{-P}$ as follows:

$$W_P(t) \sim (1 - 2^P t)^{\frac{P^2-3}{2}}, \qquad (5.117)$$

$$S_P(t) \sim (1 - 2^P t)^{\frac{1}{4}P(P-1)-1}. \qquad (5.118)$$

Further a confluent logarithmic term must occur when the exponents are positive integers, that is for watermelons when P is odd and for stars when $P = 0 \pmod 3$. The logarithm may be raised to some power. Fisher's analysis has been explicitly confirmed in the case of vicious walkers [4], while for the other models these results are confirmed by numerical analysis of the data. This numerical analysis also agrees with our exact results in the case of two and three walkers.

No. of walkers	model	stars				watermelons			
		pos.	exp.	pos.	exp.	pos.	exp.	pos.	exp.
2	for all models	1/4	−1/2	–	–	1/4	1/2	–	–
3	vicious	1/8	1/2	–	–	1/8	3	−1	3
	osculating	1/8	1/2	−1	−1	1/8	3	−1	−1, 2
	2-friendly	1/8	1/2	$-1/4 \pm 1/2i$	−1/2	1/8	3	$-1/4 \pm 1/2i$	1/2
	∞-friendly	1/8	1/2	−1	3	1/8	3	−1	3
4	vicious	1/16	2	–	–	1/16	13/2	−1/4	13/2
	osculating	1/16	2	−1/4	13/2	1/16	13/2	−1/4	13/2
	2-friendly	1/16	2			1/16	13/2		

Table 1: The numerically determined singularities. For those marked "–" there are no additional singularities. Those left blank are unidentified. The first pair of sub-columns in the star column give the position (pos.) of the dominant singularity and its exponent (exp.). The second pair of sub-columns shows the sub-dominant singularities by giving position and exponent. The singularities for the watermelons are given in analogous fashion.

A further interesting observation based on our numerical analysis relates to other (non-physical) singularities in the complex plane. We find that the non-physical singularities for three walker configurations in the osculating walker and ∞-friendly-walk model, for watermelons and stars, and for the vicious walker model for watermelons occur at the same location. Further, for the same model the exponent of the non-physical singularity is the same for stars and for watermelons, but differs from model to model. This pattern continues for configurations with four walkers in the osculating walker model, both for stars and watermelons, and for watermelons in the vicious walker model. The 2-friendly-walker model with three walkers does not fit into this pattern. It would be interesting to generate sufficient data for the four walker 2-friendly- and ∞-friendly-walk model to find sub-dominant singularities to see whether the observed behaviour for three walkers continues for four walkers. An additional interesting question is the distribution of the non-physical singularities for those models which are not associated with a vertex model, that are the n-friendly-walk models with $n \geq 3$.

6 Three and more dimensions

We know from earlier analysis [4] that three is the so called upper critical dimension for the directed walker networks. That is to say, above three dimensions the freedom to move afforded by the increased spatial dimensionality is such that the constraints imposed by the different models is not felt. (At the critical dimension the dominant singularity is usually modified by a confluent logarithmic term. Essam and Guttmann saw in [4] that this was the case for vicious walker stars and watermelons).

¿From an alternative viewpoint, the analysis of the vicious walker case [4] shows that in more than three dimensions the walkers behave like free walkers. This must apply to the other models too since the vicious walker configurations are a subset of the ones allowed in these models. All models are of course restrictions of the free walker model. Accordingly, the free walker model was analysed in [4] for three and more dimensions.

Since the mutual avoiding constraint is missing in the free walker model, and the walks in stars can end anywhere, the number of stars of length L with P walkers is given by k^{PL} where k is the number of bonds which are directed away from each site. Thus the stars have a singularity at $t = 1/k^P$ with exponent -1.

The results for $d = 3$ on the body-centred-cubic lattice for $2, 3$ and 4 walker watermelons are given in [4], where it is also proved that for all P and d the generating function is holonomic. For two free walkers in three dimensions there is a singularity at $t = 1/16$ with logarithmic corrections. In three dimensions for three walker watermelons the singularities $t = 1/64$, 1 and $-1/8$ were found, all with the exponent -1. For four walker watermelons the singularities are at $t = 0$, ∞, $1/16$, $1/256$ and $-1/64$. Again they all have the same exponent, -2. These results are given in [4] where additionally differential equations and recurrences for these models are discussed.

7 Inversion relations

To provide further insight into these models, we introduce the concept of inversion relations. This concept applies to generating functions of two or more variables [17, 1]. An inversion relation is a relation of the form

$$\begin{aligned} G(x_1, \dots, x_n) + f_1(x_1, \dots, x_n)\, G(g_1(x_1, \dots, x_n), \dots, g_n(x_1, \dots, x_n)) \\ = f_2(x_1, \dots, x_n), \end{aligned} \tag{7.119}$$

usually with relative simple functions f_1, f_2 and g_i. Typically one of the g_i is of the form $g_i = \hat{g}_i/x_i$, (hence the name *inversion* relation) with $\hat{g}_i$ a constant or a polynomial independent of x_i.

The fact that inversion relations can be found *without knowing the generating function* makes them valuable for the analysis of multi-variable series.

There are two main ways to find inversion relations. The first one is to find a bijection of the graphs onto themselves which can be translated into an inversion relation. The second way is to find an inversion relation experimentally from the first terms of the series. In the latter case the inversion relation is then only conjectural.

The great value of this concept is that in some cases one is able to determine iteratively all terms in the series and partial generating functions. That is to say, to implicitly solve the underlying problem.

In order to demonstrate these ideas, let us look in detail at the procedure in the case of two vicious walker watermelons, which of course is already solved. We then repeat the procedure to obtain the inversion relation for three vicious walker watermelons. Finally we give the general inversion relation for vicious walker watermelons which holds for any number of walks.

In this section we denote the generating functions for P vicious walker watermelons with Pj SE steps by $G_{P,k}(x) = \sum_{j=0}^{\infty} w_{P,0,j,k}\, x^j$.

The first generating functions $G_{2,k}(x)$ for 2-walker vicious watermelons with $2k$ SE steps are

$$G_{2,0}(x) = 1/(1-x^2), \tag{7.120}$$

$$G_{2,1}(x) = 1/(1-x^2)^3, \tag{7.121}$$

$$G_{2,2}(x) = (1+x^2)/(1-x^2)^5, \tag{7.122}$$

$$G_{2,3}(x) = (1+3x^2+x^4)/(1-x^2)^7, \tag{7.123}$$

$$G_{2,4}(x) = (1+6x^2+6x^4+x^6)/(1-x^2)^9, \tag{7.124}$$

$$G_{2,5}(x) = (1+10x^2+20x^4+10x^6+x^8)/(1-x^2)^{11}. \tag{7.125}$$

First we see that the numerators are symmetric, unimodal, even polynomials. Further the degree of the numerator of $G_{2,k}(x)$ is $2k-2$ for $k \geq 1$. Assuming this pattern for the $G_{2,k}$ persists, we see that the number of coefficients we need to determine is $\lfloor (k+1)/2 \rfloor$ for each $G_{2,k}$, $k \geq 1$. The series expansion for every $G_{2,k}$ gives a new coefficient for every subsequent $G_{2,i}$ with $i > k$ by using the x–y symmetry of the generating function. This implies that, for every $G_{2,k}$, $k \geq 1$, we have k constraints, although we need only $\lfloor (k+1)/2 \rfloor$ to determine the coefficients.

To find the inversion relation we are going to exploit the symmetry of the numerator. For this we replace x by $1/x$ in the $G_{2,k}(x)$ and multiply $G_{2,k}(1/x)$ by $1/x^{2k+4}$, where $-(2k+4)$ is the degree of the numerator of $G_{2,k}(x)$ minus the degree of its denominator. This maps the $G_{2,k}(x)$ onto $-G_{2,k}(x)$, with the exception of $G_{2,0}(x)$. Summing $G_{2,k}(x)y^{2k} - G_{2,k}(1/x)y^{2k}/(x^{2k+4})$ over all $k \geq 0$ directly gives the inversion relation

$$W_{2,0}(x,y) + \frac{1}{x^4} W_{2,0}(\frac{1}{x}, \frac{y}{x}) = -\frac{1}{x^2}. \tag{7.126}$$

We have seen that in this case we have more information than needed. The number of independent coefficients in the numerator could grow as k and we could still determine all $G_{2,k}$. This turns out to be the case with three walker watermelons.

To find the three vicious walker watermelon inversion relation we again take the partial generating functions $G_{3,k}(x)$ and replace x by $1/x$ and multiply the partial generating functions by $1/x^{3k+9}$, where the exponent is again determined by the difference of the degrees of the numerator and denominator. This maps $G_{3,k}(x)$ onto itself for k odd and onto $-G_{3,k}(x)$ for k even, $k > 0$. Summing this over all $k \geq 0$ and taking care of the alternating sign again gives the inversion relation

$$W_{3,0}(x,y) + \frac{1}{x^9} W_{3,0}(\frac{1}{x}, -\frac{y}{x}) = -\frac{1+x^3}{x^6}. \tag{7.127}$$

The inhomogeneous terms is caused again by the lowest order term, $G_{3,0}$.

A similar analysis allows us to identify inversion relations for the P-walker vicious watermelons. The inversion relations reflect the symmetry in the numerator polynomials of the generating functions. We observed the symmetry in the numerator already for the 2 walker case in this Section and for the 3 walker case in Section 4.2.2, and further analysis shows that it holds for any number of walkers. We find that for any number n of walkers the relation

$$W_{P,0}(x,y) + \frac{1}{x^{P^2}} W_{P,0}(\frac{1}{x},(-1)^P \frac{y}{x}) = -x^{P(1-P)} \sum_{i=0}^{P-2} x^{iP} \tag{7.128}$$

holds. The inversion relation was obtained by observation. It has not been proved. The term on the right-hand side arises since the degree of the numerator of $G_{P,0}$ does not fit into the pattern of the other generating functions, as seen in the two walker case.

As remarked above, this inversion relation, together with with the x–y symmetry of the generating function allows one to calculate the three vicious walker generating functions iteratively without the need for further data. An analysis of the partial generating functions for more than three walkers shows that the number of coefficients in the generating functions for the P-walker case increases by $P-1$ for each additional step. This implies that we cannot calculate the partial generating functions iteratively for four or more walkers without continuous input of data or a further inversion relation. The iterative procedure calculates $G_{P,(k+1)}$, assuming that we have $G_{P,i}$ for $i = 0 \dots k$. We obtain the first $k+1$ coefficients in the numerator of $G_{P,(k+1)}$ from the x–y symmetry using the $G_{P,i}$, for $i \leq k$ and up to a further $k+1$ coefficients in $G_{P,(k+1)}$ are determined by the inversion relation, or in other words by the symmetry of the numerator polynomial. The use of the symmetry of the numerator polynomial implies that this procedure successfully gives all the coefficients when at least half of them are known. This procedure implies that one can iteratively construct all partial generating functions, as long as the degree of the numerator of $G_{P,k}$ is less or equal to $2Pk$. Further, if the degree is $2Pk+c$, where c is a constant, it is likely that one can find constraints that determine a constant number of terms, such as the sum of coefficients.

One final *caveat* is that when one compares the inversion relation with the known closed form, one has to be careful in the choice of the branch of the square-root in the inverted function. For example, in the case of 2-walker vicious watermelons, one must take the negative branch. Alternatively one can write the generating function as

$$W_{2,0}(x,y) = \frac{2}{1 - x^2 - y^2 + (1-x^2)\sqrt{1 + \frac{-2y^2+y^4-2x^2y^2}{(1-x^2)^2}}}. \tag{7.129}$$

In this form the positive branch of the root has to be taken before and after the inversion.

8 Conclusion

We have introduced a family of lattice path problems, and solved a number of them. Those that were not solved have been studied numerically, and their singularity structure conjectured. Higher dimensional analogues have also been studied. A combination of exact and numerical study allows us to conjecture – and in some instances prove – the asymptotic behaviour of star and watermelon generating functions for any number of walkers (P), for all models (n), and all lattice dimensions (d). We find for square and hypercubic lattices:

$$W_{P,n}^{(2)}(t) \sim (1-2^P t)^{(P^2-3)/2} \qquad \text{for all } n,\ d=2 \tag{8.130}$$

$$S_{P,n}^{(2)}(t) \sim (1-2^P t)^{\frac{1}{4}P(P-1)-1} \qquad \text{for all } n,\ d=2 \tag{8.131}$$

$$W_{P,n}^{(3)}(t) \sim \frac{[\ln(1-3^P t)]^{P(1-P)/2}}{(1-3^P t)} \qquad \text{for all } n,\ d=3 \tag{8.132}$$

$$S_{P,n}^{(3)}(t) \sim (1-3^P t)^{2-P}[\ln(1-3^P t)]^{P(1-P)} \qquad \text{for all } n,\ d=3 \tag{8.133}$$

where the powers of the confluent logarithmic term are given in [4] eqn.(87). For $d > 3$ we have

$$W_{P,n}^{(d)}(t) \sim (1-d^P t)^{-1} \qquad \text{for all } n,\ d>3 \tag{8.134}$$

$$S_{P,n}^{(d)}(t) \sim (1-d^P t)^{1-(d-1)(P-1)/2} \qquad \text{for all } n,\ d>3 \tag{8.135}$$

which follows from [4] eqn.(3), and our observation that for $d \geq 3$ the walk configurations behave like free walkers.

As far as exact solutions go, a continuation of the anisotropic three walker analysis seems unlikely to be fruitful unless one can find an inversion relation. Similar remarks apply to the situation with four or more walkers. Other than finding exact solutions, there are few remaining questions for this family of lattice path problems.

Numerically, it should be straightforward to extend the table of singularities both to more walkers and to three or more consecutive sites, although the calculation of sufficient coefficients in the series might be excessively time consuming. It would be interesting to see if the pattern in the sub-dominant singularities which we observed in section 5 persists for more then four walkers.

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