Some solvable, and as yet unsolvable, polygon and walk models

Anthony J. Guttmann

ARC Centre of Excellence for Mathematics and Statistics of Complex Systems, Department of Mathematics and Statistics, The University of Melbourne, Victoria 3010, Australia

E-mail: `tonyg@ms.unimelb.edu.au`

Abstract. One partly solvable and two solvable models of polygons are discussed. Using a simple transfer matrix approach Iwan Jensen has derived very long series expansions for the perimeter generating function of both *three-choice polygons* and *punctured staircase polygons.* In both cases it is found that all the terms in the generating function can be reproduced from a linear Fuchsian differential equation of order 8. We report on an analysis of the properties of the differential equations. Recently Enrica Duchi has discussed the problem of so-called *prudent self-avoiding walks.* We discuss the polygon analogue of this problem, and argue that the generating function for *prudent polygons* is unlikely to be differentiably finite, though a restricted version of the problem, called prudent polygons of the second type, is likely to be differentiably finite. The exact generating function for prudent polygons of the first type is also found.

1. Introduction

A well-known long standing problem in combinatorics and statistical mechanics is to find the generating function for self-avoiding polygons (or walks) on a two-dimensional lattice, enumerated by perimeter. Recently, we have gained a greater understanding of the difficulty of this problem, as Rechnitzer [17] has *proved* that the (anisotropic) generating function for square lattice self-avoiding polygons is not differentiably finite [18], as had been previously *conjectured*, on numerical grounds [7], but not proved. That is to say, it cannot be expressed as the solution of an ordinary differential equation with polynomial coefficients. There are many simplifications of this problem that are solvable [1], but all the simpler models impose an effective directedness or other constraint that reduces the problem, in essence, to a one-dimensional problem.

One model, that of so-called *three-choice polygons*, has remained unsolved despite the knowledge that its solution must be D-finite. Recent numerical work by Guttmann and Jensen [9] has resulted in an exact differential equation apparently satisfied by the perimeter generating function of three-choice polygons.

A similar situation holds for another model, that of staircase polygons which contain an arbitrary staircase puncture. The perimeter is the sum of the internal and external perimeters, and the internal polygon has no vertices in common with the external polygon. Again it is found [10] that the perimeter generating function is apparently satisfied by an exact differential equation. While these results do not constitute a rigorous mathematical proof, the numerical evidence is overwhelmingly compelling.

The third model considered here is that of *prudent polygons.* Prudent self-avoiding walks are defined [4] as self-avoiding walks which never take a step which, if continued arbitrarily, would hit a previously visited vertex. As a consequence, the last vertex lies on the perimeter of the bounding rectangle. A typical prudent walk is shown in figure 5. Prudent polygons are just a subset of prudent walks whose end-points are adjacent to their starting points, as shown in figure 6. While we have been unable to obtain the generating function for prudent SAW or prudent polygons, we report below on some exact results for a subset of prudent polygons, and speculate on the analytic nature of the generating function.

The next three sections consider the three models described above, in turn.

2. Three-choice polygons

Three-choice self-avoiding walks on the square lattice, $\mathbb{Z}^2$, were introduced by Manna [15] and can be defined as follows: Starting from the origin one can step in any direction; after a step upward or downward one can head in any direction (except backward); after a step to the left one can only step forward or head downward, and similarly after a step to the right one can continue forward or turn upward. Alternatively put, one cannot make a right-hand turn after a horizontal step. Whittington [20] showed that the growth constant for three-choice walks is exactly 2, so that if w_n denotes the number of such walks of n steps on an infinite lattice, equivalent up to a translation, then $w_n \sim 2^{n+o(n)}$. It is perhaps surprising that the best known result for the sub-dominant term is $2^{o(n)}$ but attempts to improve on this have not been successful. Even numerically, there is no firmly based conjecture for the sub-dominant term, unlike for ordinary self-avoiding walks, for which the sub-dominant term is widely believed to be $O(\log n)$.

As usual one can define a polygon version of the walk model by requiring the walk to return to the origin. So a three-choice polygon [12] is simply a three-choice self-avoiding walk which returns to the origin, but has no other self-intersections. There are two distinct classes of three-choice polygons. The three-choice rule either leads to staircase polygons or *imperfect staircase polygons* [3] as illustrated in figure 1. In the case of staircase polygons any vertex on the perimeter can act as the origin of the three-choice walk (which then proceeds counter-clockwise), while for imperfect staircase polygons there is only one possible origin but the polygon could be rotated by 180 degrees. If we denote by t_n the number of three-choice polygons with perimeter $2n$ then, $t_n = 2nc_n + 2p_n$, where c_n is the number of staircase polygons with perimeter $2n$, and p_n is the number of imperfect staircase polygons of perimeter $2n$. Note that t_n, p_n and c_n all grow like 4^n and in particular we recall the well-known result that $c_{n+1} = C_n = \frac{1}{n+1}\binom{2n}{n}$ where C_n are the Catalan numbers.

In this paper we report on recent work [9] which has led to an exact Fuchsian [13] linear differential equation of order 8 apparently satisfied by the perimeter generating function, $\mathcal{T}(x) = \sum_{n\geq 0} t_n x^n$, for three-choice polygons (that is $\mathcal{T}(x)$ is conjectured to be one of the solutions of the ODE, expanded around the origin). The first few terms in the generating function are

$$\mathcal{T}(x) = 4x^2 + 12x^3 + 42x^4 + 152x^5 + 562x^6 + \cdots.$$

If we distinguish between steps in the x and y direction, and let $t_{m,n}$ denote the number of three-choice polygons with $2m$ horizontal steps and $2n$ vertical steps, then the anisotropic generating function for $\mathcal{T}$ can be written

$$\mathcal{T}(x,y) = \sum_{m,n} t_{m,n} x^m y^n = \sum_n H_n(x) y^n,$$

where $H_n(x) = \frac{R_n(x)}{S_n(x)}$ is the (rational [19]) generating function for three-choice polygons with $2n$ vertical steps. In earlier, unpublished, numerical work, we found that, for imperfect staircase

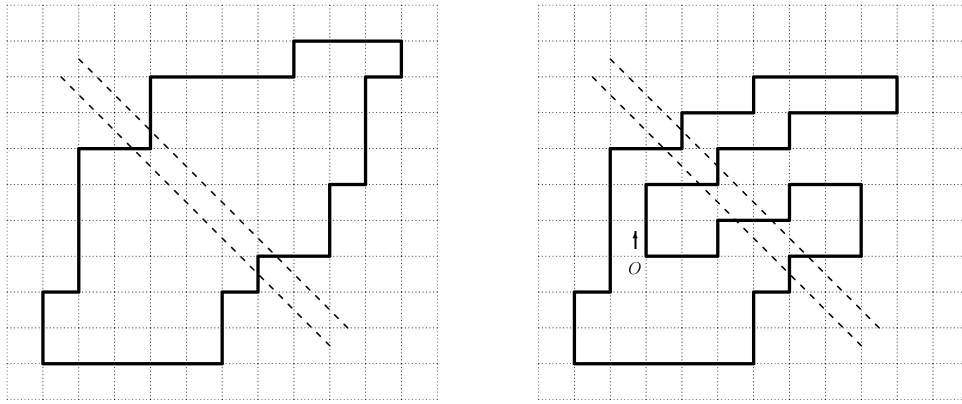


Figure 1. Examples of the two types of three-choice polygons. In the right panel we indicate the origin (O) and the direction of the first step (note that rotation by 180 degrees also leads to a valid three-choice polygon).

polygons, the denominators were:

$$S_n(x) = (1-x)^{2n-1}(1+x)^{(2n-7)_+} \qquad n \quad \text{even},$$

and

$$S_n(x) = (1-x)^{2n-1}(1+x)^{(2n-8)_+} \qquad n \quad \text{odd}.$$

This was subsequently proved by Bousquet-Mélou [2]. Further, Bousquet-Mélou showed that the numerators satisfied:

$$R_n(-1) = \frac{-12(4m)!}{m!(m+1)!(m+2)!(m+3)!} \qquad n = 2m+4,$$

and

$$R_n(-1) = \frac{-96(4m+1)!}{m!(m+1)!(m+2)!(m+4)!} \qquad n = 2m+5.$$

Unfortunately, we still do not have enough information to identify the numerators, though we observe that they are of degree $3n-7$ for $n \geq 4$ and n even, and of degree $3n-8$ for $n \geq 5$ and n odd.

It is also possible to express the generating function $\mathcal{T}(x)$ as a five-fold sum, with one constraint [2], of 4×4 Gessel-Viennot determinants [6]. This is clear from figure 2, where the enumeration of the lattice paths between the dotted lines is just the classical problem of 4 non-intersecting walkers, and these must be joined to two non-intersecting walkers to the left, and to two non-intersecting walkers to the right. Then one must sum over different possible geometries. However the fact that the generating function is so expressible implies that it is differentiably finite [14].

In the following we discuss the work leading to an ODE for the perimeter generating function of three-choice polygons. In [9] the counts for three-choice polygons up to half-perimeter 260 were generated. Using numerical experimentation what is believed to be the underlying ODE was then found. This calculation required the use of the first 206 coefficients with the resulting ODE then correctly predicting the next 54 coefficients. While the possibility that this ODE is not the correct one is extraordinarily small, this does not of course constitute a proof. Unfortunately

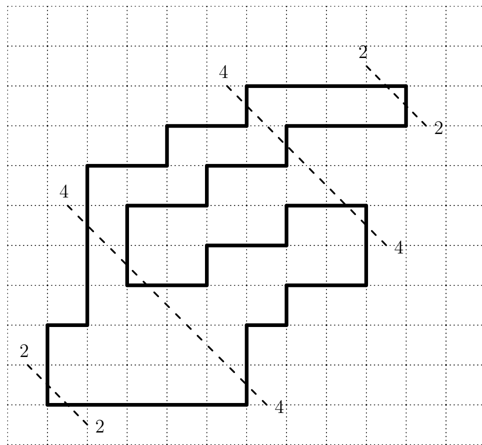


Figure 2. Showing the decomposition of an imperfect staircase polygon into a sequence of 2-4-2 non-intersecting walkers, each expressible as a Gessel-Viennot determinant

we cannot usefully bound the size of the underlying ODE, otherwise we could use the knowledge of D-finiteness to provide a proof. That is to say, any bounds that follow from closure theorems [14] are too large to be useful.

The algorithm used to count the number of imperfect polygons is a slightly modified version of the algorithm of Conway *et al.* [3], and is described fully in [9].

2.1. The Fuchsian differential equation

In recent papers Zenine *et al.* [21, 22, 23] obtained the linear differential equations whose solutions give the 3- and 4-particle contributions $\chi^{(3)}$ and $\chi^{(4)}$ to the Ising model susceptibility. In [9] their method was used to find a linear differential equation which has as a solution the generating function $\mathcal{T}(x)$ for three-choice polygons, which involves a systematic search for a differential equation of the form:

$$\sum_{k=0}^{m} P_k(x) \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathcal{T}(x) = 0, \tag{1}$$

such that $\mathcal{T}(x)$ is a solution to this homogeneous linear differential equation, where the $P_k(x)$ are polynomials. In order to make it as simple as possible they started by searching for a Fuchsian [13] equation. Such equations have only regular singular points.

They searched systematically for solutions by varying m and q_m, the degree of the polynomials $P_m(x)$. In this way a solution with $m = 10$ and $q_m = 12$ was first found, which required the determination of $L = 206$ unknown coefficients. With 260 terms in the half-perimeter series, there are more than 50 additional terms with which to check the correctness of this solution. Having found this conjectured solution the ODE was then turned into a recurrence relation and used to generate more series terms in order to search for a lower order Fuchsian equation. The lowest order equation found was eighth order and with $q_m = 30$, which requires the determination of $L = 321$ unknown coefficients. Thus from the original 260 term series this 8^{th} order solution could not have been found. This raises the question as to whether perhaps there is an ODE of lower order than 8 that generates the coefficients? The short answer to this is no.

So the (half)-perimeter generating function $\mathcal{T}(x)$ for three-choice polygons is conjectured to be a solution of the linear differential equation of order 8

Table 1. Critical exponents for the regular singular points of the Fuchsian differential equation satisfied by $\mathcal{T}(x)$.

Singularity	Exponents
$x = 0$	$-1,\ 0,\ 0,\ 0,\ 1,\ 2,\ 3,\ 4$
$x = 1/4$	$-1/2,\ -1/2,\ 0,\ 1/2,\ 1,\ 3/2,\ 2,\ 3$
$x = -1/4$	$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 13/2$
$x = \pm\,\mathrm{i}/2$	$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 13/2$
$1 + x + 7x^2 = 0$	$0,\ 1,\ 2,\ 2,\ 3,\ 4,\ 5,\ 6$
$x = \infty$	$-2,\ -3/2,\ -1,\ -1,\ -1/2,\ 1/2,\ 3/2,\ 5/2$
$Q_8(x) = 0$	$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8$

$$\sum_{k=0}^{8} P_k(x) \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) = 0 \tag{2}$$

with

$$\begin{aligned}
&P_8(x) = x^3(1-4x)^4(1+4x)(1+4x^2)(1+x+7x^2)Q_8(x),\\
&P_7(x) = x^2(1-4x)^3 Q_7(x), \qquad P_6(x) = 2x(1-4x)^2 Q_6(x),\\
&P_5(x) = 6(1-4x)Q_5(x), \quad P_4(x) = 24Q_4(x),\\
&P_3(x) = 24Q_3(x), \quad P_2(x) = 144x(1-2x)Q_2(x),\\
&P_1(x) = 144(1-4x)Q_1(x),\ P_0(x) = 576Q_0(x),
\end{aligned} \tag{3}$$

where $Q_8(x)$, $Q_7(x)$, …, $Q_0(x)$, are polynomials of degree 25, 31, 32, 33, 33, 32, 29, 29, and 29, respectively. The polynomials are given in [9].

The singular points of the differential equation are given by the roots of $P_8(x)$. One can easily check that all the singularities (including $x = \infty$) are *regular singular points* so equation (2) is indeed of the Fuchsian type. It is thus possible, using the method of Frobenius, to obtain from the indicial equation the critical exponents at the singular points. These are listed in Table 1.

A careful local analysis revealed that near the physical critical point $x = x_c = 1/4$ the singular behaviour is

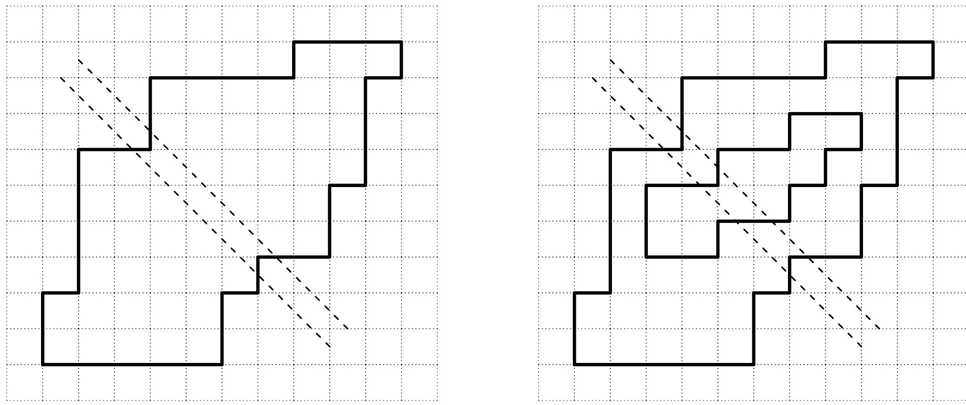
$$\mathcal{T}(x) \sim A(x)(1-4x)^{-1/2} + B(x)(1-4x)^{-1/2}\log(1-4x), \tag{4}$$

where $A(x)$ and $B(x)$ are analytic in the neighbourhood of x_c. Note that the terms associated with the exponents 1/2 and 3/2 become part of the analytic correction to the $(1-4x)^{-1/2}$ term. Near the singularity on the negative x-axis, $x = x_- = -1/4$ the singular behaviour is

$$\mathcal{T}(x) \sim C(x)(1+4x)^{13/2}, \tag{5}$$

where again $C(x)$ is analytic near x_-. Similar behaviour is expected near the pair of singularities $x = \pm\mathrm{i}/2$, and finally at the roots of $1 + x + 7x^2$ one expects the behaviour $\mathcal{T}(x) \sim D(x)(1+x+7x^2)^2 \log(1+x+7x^2)$.

To analyse the asymptotic behaviour of the coefficients, we first transform the coefficients so that the critical point is at 1. The growth constant of staircase and imperfect staircase polygons is 4, so we consider a new series with coefficients r_n, defined by $r_n = t_{n+2}/4^n$. Thus the generating function studied is $\mathcal{R}(y) = \sum_{n\geq 0} r_n y^n = 4 + 3y + 2.625y^2 + \cdots$. From equations



Staircase **Punctured staircase**

Figure 3. Examples of the types of polygons studied in this paper.

(4) and (5) it follows that the asymptotic form of the coefficients is

$$[y^n]\mathcal{R}(y) = r_n = \frac{1}{\sqrt{n}} \sum_{i \geq 0} \left(\frac{a_i \log n + b_i}{n^i} + (-1)^n \left(\frac{c_i}{n^{7+i}} \right) \right) + \mathrm{O}(\lambda^{-n}). \tag{6}$$

The last term includes the effect of other singularities, further from the origin than the dominant singularities. These will decay exponentially since $\lambda > 1$ in the scaled variable $y = x/4$.

Using the recurrence relations for t_n (derived from the ODE) it is easy and fast to generate many more terms r_n. In [9] the first 100000 terms were generated and saved as floating point numbers with 500 digit accuracy (this calculation took less than 15 minutes). With such a long series it is possible to obtain accurate numerical estimates of the first 20 amplitudes a_i, b_i, c_i for $i \leq 19$ with a precision of more than 100 digits for the dominant amplitudes, shrinking to 10–20 digits for the the case when $i = 18$, or 19. In making these estimates the exponentially decaying terms were ignored. In this way an earlier conjecture [3] that $a_0 = \frac{3\sqrt{3}}{\pi^{3/2}}$, was confirmed. Other amplitude estimates include $b_0 = 3.17327538458989848 1765\ldots$ and $c_0 = \frac{-24}{\pi^{3/2}}$, though no one has been able to identify b_0. However, further sub-dominant amplitudes have been estimated [9], such as $a_1 = \frac{-89}{8\sqrt{3}\pi^{3/2}}$, $a_2 = \frac{1019}{384\sqrt{3}\pi^{3/2}}$, and $a_3 = \frac{-10484935}{248832\sqrt{3}\pi^{3/2}}$, and $c_1 = \frac{225}{\pi^{3/2}}$, $c_2 = \frac{-16575}{16\pi^{3/2}}$, and $c_3 = \frac{389295}{128\pi^{3/2}}$. It seems likely that the amplitudes $\pi^{3/2}\sqrt{3}a_i$ and $\pi^{3/2}c_i$ are rational.

The *area* generating function is also of interest. We expect this to involve q-series, and in [9] it is found that the area generating function $A(q)$ is of the form

$$A(q) = (G(q) + H(q)/\sqrt{1 - q/\eta})/[J_0(1,1,q)^2],$$

where $J_0(x,y,q)$ is a q-generalisation of the Bessel function, and occurs, for example, in the solution of the problem of staircase polygons enumerated by area [1]. Here $q = \eta$ is the first zero of $J_0(1,1,q)$, and G and H are regular in the neighbourhood of $q = \eta$. The coefficients thus behave asymptotically as

$$a_n = [q^n]A(q) \sim const.\eta^{-n} n^{3/2}.$$

3. Punctured staircase polygons

A staircase polygon can be viewed as the intersection of two directed walks starting at the origin, moving only to the right or up and terminating once the walks join at a vertex. It is well-known that the perimeter generating function for staircase polygons is

$$P(x) = \frac{1-2x-\sqrt{1-4x}}{2} \propto (1-\mu x)^{2-\alpha},$$

where the connective constant $\mu = 4$ and the critical exponent $\alpha = 3/2$. The coefficients in the expansion of $P(x)$ are just the Catalan numbers.

Punctured staircase polygons [8] are staircase polygons with internal holes which are also staircase polygons (the polygons are mutually- as well as self-avoiding). In [8] it was proved that the connective constant μ of k-punctured polygons (polygons with k holes) is the same as the connective constant of un-punctured polygons. Numerical evidence clearly indicated that the critical exponent α_k increased by 3/2 per puncture. Here we discuss only the case with a *single* hole (see figure 3), and we refer to these objects as punctured staircase polygons. The perimeter length of staircase polygons is even and thus the total perimeter (the outer perimeter plus the perimeter of the hole) is also even. We denote by p_n the number of punctured staircase polygons of total perimeter $2n$. The results of [8] indicate that the half-perimeter generating function has a simple pole at $x = x_c = 1/\mu = 1/4$, though the analysis in [8] clearly indicated that the critical behaviour is more complicated.

Here we report on recent work [10] which has led to an exact Fuchsian linear differential equation of order 8 apparently satisfied by the perimeter generating function, $\mathcal{P}(x) = \sum_{n\geq 0} p_n x^n$, for punctured staircase polygons (that is $\mathcal{P}(x)$ is one of the solutions of the ODE, expanded around the origin). The first few terms in the generating function are

$$\mathcal{P}(x) = x^8 + 12x^9 + 94x^{10} + 604x^{11} + 3463x^{12} + \cdots.$$

The situation is very similar to that of three-choice polygons, discussed above. This is perhaps not surprising, as one can represent punctured staircase polygons as the the fusion of two three-choice polygons, with some edges deleted. Our analysis of the ODE shows that the dominant singular behaviour is

$$\mathcal{P}(x) = A(x)(1-4x)^{-1} + B(x)(1-4x)^{-1/2} + C(x)(1-4x)^{-1/2}\log(1-4x) + D(x)(1+4x)^{13/2}. \quad (7)$$

As for three-choice polygons, it is possible to express the generating function $\mathcal{P}(x)$ of punctured staircase polygons as a sum over 4×4 Gessel-Viennot determinants. This is clear from figure 4. By arguments similar to those presented above, it follows that the generating function is D-finite.

As for three-choice polygons, we cannot readily bound the size of the underlying ODE, otherwise we could use this observation to provide a proof of our results. However, from the counts of the first 260 polygons (up to perimeter 520), the underlying ODE was found experimentally from the first 206 coefficients [10]. The ODE then correctly predicted the next 54 coefficients. While the possibility that the underlying ODE is not the correct one is extraordinarily small, that still does not constitute a proof.

The enumeration algorithm is, as for three-choice polygons, a modified version of the algorithm of Conway *et al.* [3] for the enumeration of imperfect staircase polygons, and is described in [10].

The ODE was identified in a manner similar to that described above for three-choice polygons, and the (half)-perimeter generating function $\mathcal{P}(x)$ for punctured staircase polygons was found to satisfy the linear differential equation of order 8

$$\sum_{k=0}^{8} P_n(x) \frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) = 0 \quad (8)$$

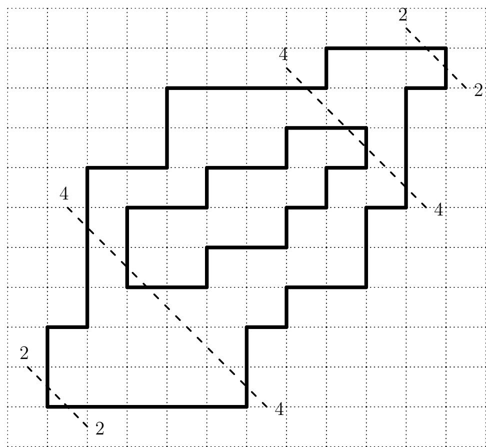


Figure 4. Showing the decomposition of a punctured staircase polygon into a sequence of 2-4-2 vicious walkers, each expressible as a Gessel-Viennot determinant

Table 2. Critical exponents for the regular singular points of the Fuchsian differential equation satisfied by $\mathcal{P}(x)$.

Singularity	Exponents
$x = 0$	$-1, 0, 0, 0, 1, 2, 3, 8$
$x = 1/4$	$-1, \;\; -1/2, -1/2, 1/2, 1, 3/2, 2, 3$
$x = -1/4$	$0, 1, 2, 3, 4, 5, 6, 13/2$
$x = \pm\,\mathrm{i}/2$	$0, 1, 2, 3, 4, 5, 6, 13/2$
$1 + x + 7x^2 = 0$	$0, 1, 2, 2, 3, 4, 5, 6$
$1/x = 0$	$-2, -3/2, -1, -1, -1/2, 1/2, 3/2, 5/2$
$Q_8(x) = 0$	$0, 1, 2, 3, 4, 5, 6, 8$

with

$$
\begin{gathered}
P_8(x) = x^4(1-4x)^8(1+4x)(1+4x^2)(1+x+7x^2)Q_8(x). \\
P_7(x) = x^3(1-4x)^7Q_7(x), \qquad P_6(x) = 2x^2(1-4x)^6Q_6(x), \\
P_5(x) = 6x(1-4x)^5Q_5(x), \qquad P_4(x) = 120(1-4x)^4Q_4(x), \\
P_3(x) = 120(1-4x)^3Q_3(x), \qquad P_2(x) = 720(1-4x)^2Q_2(x), \\
P_1(x) = 720(1-4x)Q_1(x), \qquad P_0(x) = 2880Q_0(x),
\end{gathered} \tag{9}
$$

where $Q_8(x)$, $Q_7(x)$, …, $Q_0(x)$, are polynomials of degree 22, 28, 29, 30, 31, 31, 31, 31, and 31, respectively. The polynomials are given in [10].

The singular points of the differential equation are given by the roots of $P_8(x)$. Using the method of Frobenius the critical exponents at the singular points were obtained and are listed in Table 2.

Detailed analysis of the local solutions of the ODE are given in [10]. Near the physical critical point $x = x_c = 1/4 = 1/\mu$ the following singular behaviour was found:

$$
F(x) \sim A(x)(1-4x)^{-1} + B(x)(1-4x)^{-1/2} + C(x)(1-4x)^{-1/2}\log(1-4x), \tag{10}
$$

where $A(x)$, $B(x)$ and $C(x)$ are analytic in a neighbourhood of x_c. Note that the terms associated with the exponents 1/2 and 3/2 become part of the analytic correction to the $(1-4x)^{-1/2}$ term.

Near the singularity on the negative x-axis, $x = x_- = -1/4$ the singular behaviour

$$F(x) \sim D(x)(1+4x)^{13/2}, \tag{11}$$

was found, where again $D(x)$ is analytic near x_-. Similar behaviour is expected near the pair of singularities $x = \pm \mathrm{i}/2$, and finally at the roots of $1 + x + 7x^2$ the behaviour $E(x)(1+x+7x^2)^2 \log(1+x+7x^2)$ is expected.

The asymptotic form of the coefficients is analysed just as for three-choice polygons. The critical point is transformed so as to be at 1. The growth constant of punctured staircase staircase polygons is 4, so the series was normalised by considering the new series with coefficients r_n, defined by $r_n = p_{n+8}/4^n$. Thus the generating function studied was $\mathcal{R}(y) = \sum_{n \geq 0} r_n y^n = 1 + 3y + 5.875y^2 + \cdots$. Using the recurrence relations for p_n (derived from the ODE) it is easy and fast to generate many more terms r_n. From equations (10) and (11) it follows that the asymptotic form of the coefficients is

$$[x^n]\mathcal{R}(y) = r_n = \sum_{i \geq 0} \left(\frac{a_i}{n^i} + \frac{b_i \log n + c_i}{n^{i+1/2}} + (-1)^n \left(\frac{d_i}{n^{15/2+i}} \right) \right). \tag{12}$$

Any contributions from the other singularities are exponentially suppressed since their norm (in the scaled variable $y = x/4$) exceeds 1.

From the first 100000 terms, quickly generated from the ODE, estimates for the amplitudes were obtained by fitting r_n to the form given above using an increasing number of amplitudes. Doing this led to the refined asymptotic form

$$[x^n]\mathcal{R}(y) = r_n = 1024 \left(1 + \frac{1}{\sqrt{n}} \sum_{i \geq 0} \left(\frac{b_i \log n + c_i}{n^i} + (-1)^n \left(\frac{d_i}{n^{7+i}} \right) \right) \right). \tag{13}$$

From the very long series it was possible to obtain accurate numerical estimates of many of the amplitudes b_i, c_i, and d_i, with precision of more than 100 digits for the dominant amplitudes, shrinking to around 10 digits for the the case when $i = 18$. In this way it was found that [10] $b_0 = -\frac{6\sqrt{3}}{\pi^{3/2}}$, $b_1 = \frac{305}{4\sqrt{3}\pi^{3/2}}$, $b_2 = \frac{86123}{192\sqrt{3}\pi^{3/2}}$, $c_0 = 1.55210340048879105374\ldots$ and $d_0 = \frac{48}{\pi^{3/2}}$, $d_1 = -\frac{2610}{\pi^{3/2}}$, $d_2 = \frac{640815}{8\pi^{3/2}}$, $d_3 = -\frac{116785575}{64\pi^{3/2}}$, $d_4 = \frac{70325480841}{2048\pi^{3/2}}$, though c_0 has not been identified. These amplitudes are known to at least 100 digits accuracy. The excellent convergence is solid evidence (though naturally not a proof) that the assumptions leading to equation (12) are correct. Further evidence is also advanced in [10].

As for three-choice polygons, the *area* generating function is expected to involve q-series, and appears to be of the form

$$A(q) = (G(q) + H(q)\sqrt{1 - q/\eta})/[J_0(1,1,q)^2],$$

where $J_0(x,y,q)$ is as described above. Here $q = \eta$ is the first zero of $J_0(1,1,q)$, and G and H are regular in the neighbourhood of $q = \eta$. The coefficients thus behave asymptotically as

$$a_n = [q^n]A(q) \sim const.\eta^{-n}n.$$

4. Prudent polygons

A recently proposed model [4] called *prudent* self-avoiding walks was first introduced to the mathematics community in an unpublished manuscript [16]by Préa. More recently, Duchi [4] studied the problem, and solved two proper subsets, which she called *prudent walks of the first*

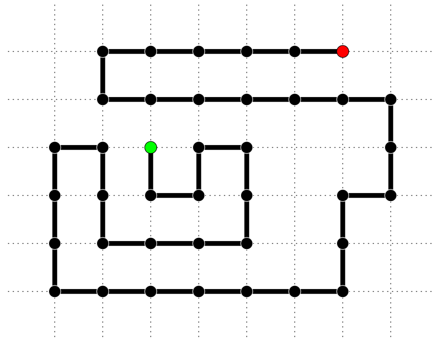


Figure 5. Example of a prudent SAW

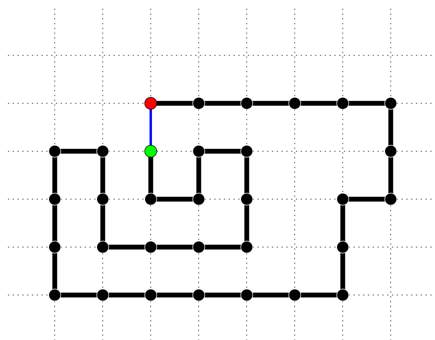


Figure 6. Example of a prudent SAP

type and *prudent walks of the second type.* The generating function in both cases was found to be algebraic. She also obtained two functional equations which could be iterated to produce the series coefficients for prudent SAW in polynomial time. However her results on prudent walks of the second type are incorrect, so the nature of that generating function remains open.

As usual one can define a polygon version of the walk model by requiring the walk to return to a site adjacent to the origin, see figure 6. Dethridge and Guttmann [5] have generated extensive isotropic and anisotropic series expansions for prudent polygons.

If we distinguish between steps in the x and y direction, and let $p_{m,n}$ denote the number of prudent polygons with $2m$ horizontal steps and $2n$ vertical steps, then the anisotropic generating function for polygons can be written

$$P(x, y) = \sum_{m,n} p_{m,n} x^m y^n = \sum_n H_n(x) y^n,$$

where $H_n(x) = \frac{R_n(x)}{S_n(x)}$ is the (rational [19]) generating function for prudent polygons with $2n$ vertical steps.

In [5] a transfer-matrix algorithm was used to count the number of prudent SAWs and polygons. The basis of the algorithm is that if we are given a walk that is a prefix of a prudent SAW or polygon, we can determine in how many ways that walk can be extended to form a prudent SAW or polygon with only a small amount of information about the walk. This information is called a *configuration.* All SAWs of a given length correspond to one of a finite number of these configurations. The algorithm progresses by computing how many walks there are of each possible configuration up to a given number of steps. The information that must be stored in a configuration depends on which objects (for example, walks or polygons, and of which type) one is counting. Further details can be found in [5].

Duchi defined prudent walks of the first type as those in which a south step may not be followed by a west step, nor a west step by a south step. She then showed that the generating function for such walks was algebraic,

$$C_0(x) = \sum_n c_n^{(0)} x^n = \frac{t(1-2x-x^2)(3+2x-3x^2) + t(1-t)\sqrt{(1-t^4)(1-2t-t^2)}}{(1-2x-x^2)(1-2x-2x^2+2x^3)}.$$

From this it can be seen that $c_n^{(0)} \sim const.\mu_0^n + \mathrm{O}(1/n)$, where $1/\mu_0$ is the smallest positive root of the polynomial $1-2x-2x^2+2x^3$.

The generating function satisfies the second order linear ordinary differential equation (ODE) $P_2(x)f''(x) + P_1(x)f'(x) + P_0 f(x) = 0$, where

$$P_2(x) = -(1-x^4)(1-2x-x^2)(1-2x-2x^2+2x^3)(1+2x^3-x^4)(1-2x+x^2+4x^3+2x^4)/4$$

$$\begin{aligned} P_1(x) &= (2-x-14x^2+17x^3+12x^4-66x^5+2x^6-30x^7+44x^8+105x^9 \\ &\quad -64x^{10}-7x^{11}+18x^{12}-26x^{13}-20x^{14}+4x^{16}])/2, \end{aligned}$$

$$P_0(x) = (1-13x+16x^2+3x^3-34x^4+9x^5+60x^6-9x^7-15x^8-20x^9-28x^{10}+10x^{11}-4x^{13})$$

Duchi then defined prudent walks of the second type by the rule that a west step may not be followed by a south step when the walk visits the top of its bounding rectangle, and a west step may not be followed by a north step when the walk visits the bottom of its bounding rectangle. These rules do not apply to the degenerate case when the walk is a straight line. She found a fourth order algebraic equation for the generating function. In fact we find this to be wrong, though we have not as yet found the correct result.

Duchi also gave two coupled functional equations for the problem of prudent SAW (that is to say, the unrestricted problem), which allows the series coefficients to be calculated in polynomial time. No closed form solution for the generating function has been given

We can similarly define prudent polygons, prudent polygons of the 1st type and prudent polygons of the 2nd type as the corresponding prudent walks whose end-points are adjacent to their starting point. An additional bond then gives a polygon.

For prudent walks of the first type, we find the generating function satisfies a fourth order linear ODE,

$$\sum_{i=0}^{4} P_i(x) f^{(i)}(x) = 0,$$

where

$$P_0(x) = 4(3+4x-13x^2+10x^3),$$
$$P_1(x) = 4(11-27x+34x^2-47x^3+35x^4),$$
$$P_2(x) = (-41+164x-214x^2+136x^3-131x^4+92x^5),$$
$$P_3(x) = (1-x)(12-89x+161x^2-81x^3+29x^4-54x^5)/3,$$
$$P_4(x) = x(1-x)^2(2-3x)(1-3x-x^2-x^3)/3.$$

The the growth constant (which is the reciprocal of the critical point) is given by the real, positive zero of the cubic $1-3x-x^2-x^3$, $\mu^2 = 3.3829757...$ and the critical exponent follows from the indicial equation. The singular behaviour It implies a square root singularity for the generating function of polygons of perimeter $2n$. Denoting the number of such polygons as $p_{2n}^{(1)}$ the generating function is thus given by

$$P^{(1)}(x) = \sum_n p_{2n}^{(1)} x^n \sim A\sqrt{1-(\mu_p^{(1)})^2 x}.$$

We remark in passing that the growth constant for prudent polygons of the first type $(\mu_p^{(1)})^2 = 3.3829..$ is less than that for prudent walks of the first type, in which case $(\mu_w^{(1)})^2 = 6.15630..$ [4].

For prudent polygons of the second type, we have been unable to find the ODE satisfied by the generating function. However, by the usual techniques of series analysis, we estimate the critical point to be $(\mu_p^{(2)})^2 = 4.096158..$ and exponent again a square root, as for prudent polygons of the first type. As for prudent polygons of the first type, we note that the growth constant for prudent polygons of the second type is less than that for prudent walks of the second type, (which is the same as that for prudent walks of the first type), that is $(\mu_w^{(1)})^2 = (\mu_w^{(2)})^2 = 6.15630..$ [5].

Additionally, we generated series for *anisotropic* prudent polygons of the second type. Let $p_{m,n}$ denote the number of prudent polygons of the second type with horizontal semi-perimeter m and vertical semi-perimeter n, so that the generating function is

$$P(x,y) = \sum_{m,n} p_{m,n} x^m y^n = \sum_n H_n(x) y^n,$$

we find for $H_n(x)$, the generating function for such polygons with precisely $2n$ vertical bonds,

$$H_n = \frac{x^{n+1} P_n(x)}{(1-y^2)^{n/2}},$$

for n even, and

$$H_n = \frac{x^{n+2} P_{2n}(x)}{(1-y^2)^n}$$

for n odd.

Based on experience with other walk and polygon models, we *conjecture* that the generating function for the *isotropic* type 2 polygon case is D-finite. We have thus far been unable to find the generating function.

For prudent polygons, the corresponding denominators are:

$$H_1(x) = \frac{1+x-x^2}{(1-x)^3}$$

$$H_2(x) = \frac{1+3x+2x^2+x^3}{(1-x)^5}$$

$$H_3(x) = \frac{1+5x+x^2-3x^3+x^5}{(1-x)^5(1+x)}$$

$$H_4(x) = \frac{1+7x+16x^2+18x^3+12x^4+4x^5+x^6}{(1-x)^7(1+x)}$$

$$H_5(x) = \frac{1+11x+19x^2+3x^3-12x^4+5x^5+6x^6-x^7-x^8}{(1-x)^7(1+x)^2}$$

$$H_6(x) = \frac{V_9(x)}{(1-x)^9(1+x)^2}$$

$$H_7(x) = \frac{V_{14}(x)}{(1-x)^9(1+x)^4(1+x+x^2)}$$

$$H_8(x) = \frac{V_{15}(x)}{(1-x)^{11}(1+x)^4(1+x+x^2)},$$

where $V_i(x)$ denotes a polynomial of degree i. From the above, we see the relentless build-up of cyclotomic polynomials of increasingly high order. As is well-known, due to a theorem of Bousquet-Mélou [2], the underlying generating function cannot be D-finite if this behaviour continues. While this does not *a priori* prove that the *isotropic* generating function is not D-finite, we know of no combinatorial problem where this is the case. That is to say, where the isotropic generating function *is* D-finite, while the anisotropic generating function is not. We therefore conjecture that the isotropic generating function for both prudent SAW and prudent polygons is not D-finite.

5. Conclusion

We have described a variety of procedures, based on improved algorithms, for enumerating three choice polygons, punctured staircase polygons and prudent polygons of various types. The extended series, permit us to conjecture the exact solution in the case of three choice polygons, punctured staircase polygons and prudent polygons of the first type. We believe that the generating function for prudent polygons and walks of the second type is also D-finite, but have been unable to find it thus far. We also conjecture that the generating functions for prudent polygons, and prudent SAWs, are not D-finite

In a subsequent publication [11] we propose to investigate the area generating function more fully, and to say more about the properties of the ODE we have found for the perimeter generating function. In particular, we discuss some simple solutions of the ODE, and ask what these can tell us about the full solution.

Acknowledgments

We would like to thank Simone Rinaldi and Enrica Duchi for introducing us to the problem of prudent SAW. I would also like to thank Iwan Jensen and John Dethridge for permission to extensively quote from our joint papers, which form the basis of much of this paper. We gratefully acknowledge financial support from the Australian Research Council.

References

[1] Bousquet-Mélou M 1996 A method for the enumeration of various classes of column-convex polygons *Disc. Math.* **154** 1–25

[2] Bousquet-Mélou M Private communication.

[3] Conway A R, Guttmann A J and Delest M 1997 The number of three-choice polygons *Mathl. Comput. Modelling* **26** 51–58

[4] Duchi E, On some classes of prudent walks *Proceedings of Formal Power Series and Algebraic Combinatorics 2005, FPSAC2005*

[5] Dethridge J C and Guttmann A J Prudent self-avoiding walks and polygons *in preparation*

[6] Gessel I and Viennot X G 1989 Determinants, paths and plane partitions *preprint*

[7] Guttmann A J and Conway A R 2001 Square lattice self-avoiding walks and polygons *Ann. Comb.* **5** 319–345

[8] Guttmann A J Jensen I Wong L H and Enting I G 2000 Punctured polygons and polyominoes on the square lattice *J. Phys. A: Math. Gen.* **33** 1735-1764

[9] Guttmann A J and Jensen I Fuchsian differential equation for the perimeter generating function of three-choice polygons. *Séminaire Lotharingien de Combinatoire, in press*

[10] Guttmann A J and Jensen I Fuchsian differential equation for the perimeter generating function of punctured staircase polygons. *In preparation*

[11] Guttmann A J and Jensen I Properties of Fuchsian differential equations for polygon enumeration problems *In preparation*
[12] Guttmann A J, Prellberg T and Owczarek A L 1993 On the symmetry classes of planar self-avoiding walks *J. Phys. A: Math. Gen.* **26** 6615–6623
[13] Ince E L 1927 *Ordinary differential equations* (London: Longmans, Green and Co. Ltd.)
[14] Lipshitz L 1989 D-finite power series, *J. Algebra* **122** 353–373
[15] Manna S S 1984 Critical behaviour of anisotropic spiral self-avoiding walks *J. Phys. A: Math. Gen.* **17** L899–L903
[16] Préa P Exterior self-avoiding walks on the square lattice, *Unpublished manuscript*
[17] Rechnitzer A 2003 Haruspicy and anisotropic generating functions *Advances in Applied Mathematics* **30** 228–257
[18] Stanley R P 1980 Differentiably finite power series *European J. Combin.* **1** 175–188
[19] Stanley R P 1999 *Enumerative Combinatorics* vol. 2 (Cambridge: Cambridge University Press)
[20] Whittington S G 1985 Anisotropic spiral self-avoiding walks *J. Phys. A: Math. Gen.* **18** L67–L69
[21] Zenine N, Boukraa S, Hassani S and Maillard J M 2004 The Fuchsian differential equation of the square lattice Ising model $\chi^{(3)}$ susceptibility *J. Phys. A: Math. Gen.* **37** 9651–9668
[22] Zenine N, Boukraa S, Hassani S and Maillard J M 2005 Square lattice Ising model susceptibility: series expansion method and differential equation for $\chi^{(3)}$ *J. Phys. A: Math. Gen.* **38** 1875–1899
[23] Zenine N, Boukraa S, Hassani S and Maillard J M 2005 Ising model susceptibility: the Fuchsian differential equation for $\chi^{(4)}$ and its factorization properties *J. Phys. A: Math. Gen.* **38** 4149–4173