Electoral Equilibria under Scoring Voting Rules

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Introduction

Candidates running in an election must decide where they stand on the ideological spectrum in order to maximise the support of the voters measured by some voting rule.

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- What kind of equilibria?

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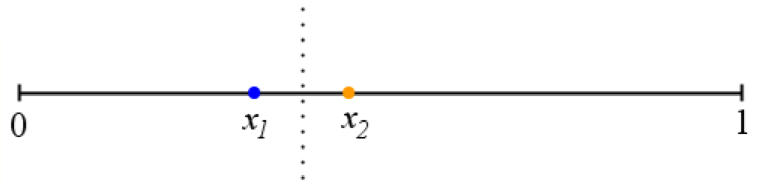

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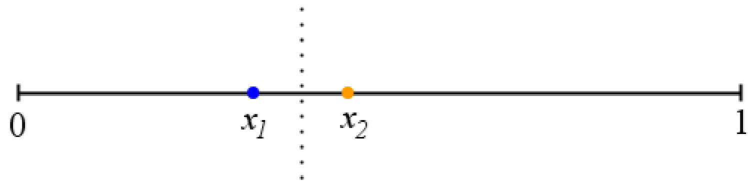

- There are m candidates. A *profile* is an m vector $x = (x_1, \ldots, x_m) \in [0, 1]^m$ that specifies each candidate's position: x_i is candidate i's position.

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- The basic theme of Myerson's Schumpeter Lecture (1998, Berlin meetings of the European Economic Association) is the importance of explicitly comparing different electoral systems in Hotelling type models.
- Myerson concentrated on positional scoring rules, we follow him in this.

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- Candidates are *score (share) maximisers*.

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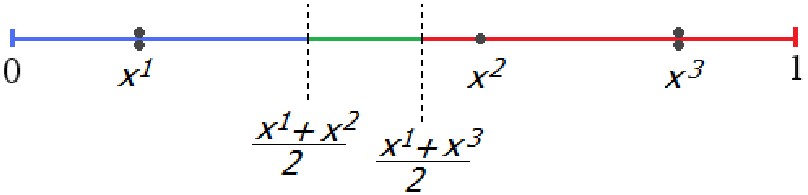
- For example, if Borda rule is used:

Ranking	Points received
A	6
B	5
$C \sim D \sim E$	$3 = \frac{1}{3}(4+3+2)$
F	1
G	0

Workings of a positional scoring rule



The score of a candidate positioned at x^1 would be

$$\frac{s_1+s_2}{2}\frac{x_1+x_2}{2}+\frac{s_2+s_3}{2}\frac{x_3-x_2}{2}+\frac{s_4+s_5}{2}\left(1-\frac{x_1+x_3}{2}\right).$$

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- A *convergent* Nash equilibrium (CNE) occurs when all candidates adopt the same ideological position.
- A *non-convergent* Nash equilibrium (NCNE) is when not all candidate positions are the same.

Convergent equilibria

Theorem (Cox, 1987). For m candidates and scoring rule s, a profile $x = (x^*, \ldots, x^*)$ is a CNE if and only if

$$c(s, m) \leq x^* \leq 1 - c(s, m), \tag{1}$$

where $c(s, m) = \dfrac{s_1 - \bar{s}}{s_1 - s_m}$ is the *c-value* (with $\bar{s} = \frac{1}{m} \sum_{i=1}^{m} s_i$) .

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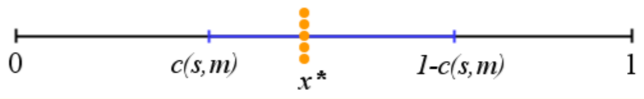

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- It is an easy observation that in a three-candidate election under any positional scoring rule no NCNE exist.
- The first question: If $m = 4$, can we characterize the rules for which NCNE exist?

The four-candidate case

Theorem (CMS., 2012). In a four-candidate election under scoring rule $s = (s_1, s_2, s_3, s_4)$, NCNE exist iff both the following conditions are satisfied:

- a) $c(s, 4) > 1/2$ (that is no CNE exist);
- b) $s_1 > s_2 = s_3$.

Moreover, the NCNE is unique and symmetric. Two paired candidates at $x_1 = \dfrac{s_1}{4(s_1 - s_2)}$ and $x_2 = 1 - x_1$.

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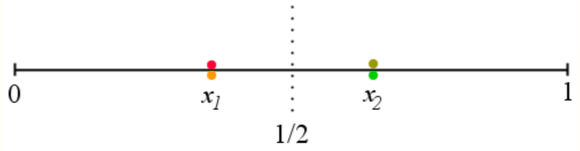

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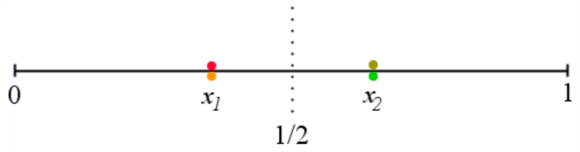

If $c(s, 4) > 1/2$ but $s_2 \neq s_3$ then no NE of either kind exist.

The five-candidate case

Theorem (CMS., 2012). In a five-candidate election under scoring rule $s = (s_1, s_2, s_3, s_4, s_5)$, NCNE exist iff both the following conditions are satisfied:

- a) $s_1 > s_2 = s_3 = s_4$;
- b) $c(s, 5) > 1/2$.

Moreover, the NCNE is unique and symmetric, with equilibrium profile $x = ((x^1, 2), (1/2, 1), (x^2, 2))$, where

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Note. For both $m = 4$ and $m = 5$ CNE and NCNE cannot coexist together. This will be broken for $m = 6$.

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Theorem (CMS., 2012). Given $m = 6$ and scoring rule $s = (s_1, s_2, s_3, s_4, s_5, s_6)$. Then there are four possible types of equilibria split in two groups:

$$\{(2, 2, 2),\ (2, 1, 1, 2)\} \text{ and } \{(3, 3), (6)\}.$$

The equilibria of the first group occur for rules s that satisfy

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The equilibria within each group can coexist but no equilibrium of the first group can coexist with an equilibrium of the second group. In particular, CNE and NCNE can coexist.

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Convex scores

We say that the score vector $s = (s_1, \dots, s_m)$ is convex if

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Theorem (CMS, 2012). Let s be a convex scoring rule $s = (s_1, \ldots, s_n, s_{n+1}, \ldots, s_m)$, with

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for some $1 \leq n < m$. Then there are no NCNE, unless the subrule $s' = (s_1, \ldots, s_n, s_{n+1})$ is Borda and $n + 1 \leq \lfloor m/2 \rfloor$ (i.e., more than half the scores are constant). In the latter case NCNE do exist.

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Example. $s = (3, 2, 1, 0, 0, 0, 0)$.

Concave and weakly concave scores

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A weakly concave rule is either worst-punishing or intermediate.

Surprising properties of weakly convex rules

Theorem (CMS, 2012). Any weakly concave scoring rule s has no NCNE

$$x = ((x^1, n_1), \ldots, (x^q, n_q))$$

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This means that if a concave rule has an NCNE it has to have more than half of all candidates in one of the extreme locations!

Such weakly convex rules exist

For $m = 12$ the scoring rule $s = (4, 4, 4, 3, 3, 3, 2, 1, 1, 0, 0, 0)$ satisfies weak concavity, yet does allow NCNE. In particular, the profile

$$((x^1, n_1), (x^2, n_2)) = \left(\left(\frac{13}{28}, 8\right), \left(\frac{41}{84}, 4\right)\right)$$

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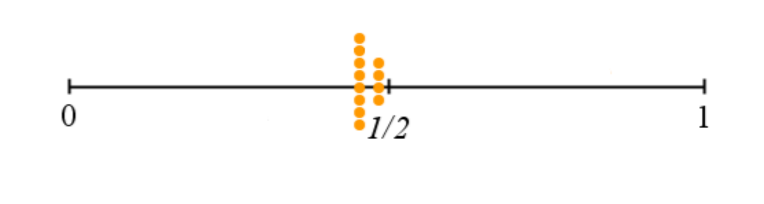

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Thanks for your attention!