Composition of Simple Games

Rupert Freeman
Supervisor: Arkadii Slinko

University of Auckland

rfre038@aucklanduni.ac.nz

March 24, 2013

Introduction

- A simple game is a pair $G = (P_G, W_G)$, where P_G is a set of players and $W_G \subseteq 2^{P_G}$ is a non-empty set of subsets (coalitions) which satisfy the monotonicity condition:

$$\text{if } X \in W_G \text{ and } X \subseteq Y, \text{ then } Y \in W_G$$

Coalitions from W_G are called *w*inning coalitions of G, the others are called *l*osing coalitions.

Introduction

- A simple game is a pair $G = (P_G, W_G)$, where P_G is a set of players and $W_G \subseteq 2^{P_G}$ is a non-empty set of subsets (coalitions) which satisfy the monotonicity condition:

  $$\text{if } X \in W_G \text{ and } X \subseteq Y, \text{ then } Y \in W_G$$

  Coalitions from W_G are called *winning* coalitions of G, the others are called *losing* coalitions.
- eg. UN Security Council
  5 permanent members, 10 non-permanent members. A vote requires the support of all 5 permanent members in addition to at least 4 non-permanent members to pass.

Introduction

- A simple game is a pair $G = (P_G, W_G)$, where P_G is a set of players and $W_G \subseteq 2^{P_G}$ is a non-empty set of subsets (coalitions) which satisfy the monotonicity condition:

  $$\text{if } X \in W_G \text{ and } X \subseteq Y, \text{ then } Y \in W_G$$

  Coalitions from W_G are called *winning* coalitions of G, the others are called *losing* coalitions.
- eg. UN Security Council
  5 permanent members, 10 non-permanent members. A vote requires the support of all 5 permanent members in addition to at least 4 non-permanent members to pass.
- $[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$

Composition of Games - Motivation

- Consider the board of a large company, who vote to make strategy decisions under a certain voting rule. Suppose one of the board members retires, but it is decided that their knowledge and experience is too great to replace with just a single person. Instead, a group of people fills the one spot on the board. They collectively vote on each issue. A collective yes vote means that the ex-board members vote is a yes, a collective no means that the ex-board members vote is a no.

Composition of Games - Motivation

- Consider the board of a large company, who vote to make strategy decisions under a certain voting rule. Suppose one of the board members retires, but it is decided that their knowledge and experience is too great to replace with just a single person. Instead, a group of people fills the one spot on the board. They collectively vote on each issue. A collective yes vote means that the ex-board members vote is a yes, a collective no means that the ex-board members vote is a no.
- What properties does the resulting voting structure (game) have?

Composition of Games - Motivation

- Consider the board of a large company, who vote to make strategy decisions under a certain voting rule. Suppose one of the board members retires, but it is decided that their knowledge and experience is too great to replace with just a single person. Instead, a group of people fills the one spot on the board. They collectively vote on each issue. A collective yes vote means that the ex-board members vote is a yes, a collective no means that the ex-board members vote is a no.
- What properties does the resulting voting structure (game) have?
- Is it possible to incorporate all voters in a one-step process, or do we require two separate votes?

Definition

Let G and H be two games such that P_G and P_H are disjoint. Define the composition $C = G \circ_g H$ via player $g \in P_G$ by $P_C = (P_G \backslash \{g\}) \cup P_H$ and

$$W_C^{min} = \{X \subseteq P_C : X \in W_G^{min}\} \cup \{X \subset P_C : (X \cap P_G) \cup \{g\} \in W_G^{min} \text{ and } X \cap P_H \in W_H^{min}\}$$

Composition of Games - Definition

Definition

Let G and H be two games such that P_G and P_H are disjoint. Define the composition $C = G \circ_g H$ via player $g \in P_G$ by $P_C = (P_G \backslash \{g\}) \cup P_H$ and

$$W_C^{min} = \{X \subseteq P_C : X \in W_G^{min}\} \cup \{X \subset P_C : (X \cap P_G) \cup \{g\} \in W_G^{min} \text{ and } X \cap P_H \in W_H^{min}\}$$

- eg. Consider the case where $G = H$ are k out of n majority games. Then the minimal winning coalitions of $G \circ_g H$ are those consisting of k players from G, or $k - 1$ players from G and k players from H.

Complete Games and Weighted Voting Games

Definition

Let $G = (P_G, W_G)$. We define the desirability relation $\preceq$ on G by: $i \preceq_G j$ if for all $U \subseteq P_G \setminus \{i, j\}$, $U \cup i \in W_G \implies U \cup j \in W_G$. We say that j is more desirable than i.

Definition

Let $G = (P_G, W_G)$. We define the desirability relation $\preceq$ on G by: $i \preceq_G j$ if for all $U \subseteq P_G \backslash \{i, j\}$, $U \cup i \in W_G \implies U \cup j \in W_G$. We say that j is more desirable than i.

- This is a partial ordering.

Complete Games and Weighted Voting Games

Definition

Let $G = (P_G, W_G)$. We define the desirability relation $\preceq$ on G by: $i \preceq_G j$ if for all $U \subseteq P_G \backslash \{i, j\}$, $U \cup i \in W_G \implies U \cup j \in W_G$. We say that j is more desirable than i.

- This is a partial ordering.
- Say that a game is complete if "$\preceq$" is a total ordering.

Complete Games and Weighted Voting Games

Definition

Let $G = (P_G, W_G)$. We define the desirability relation $\preceq$ on G by: $i \preceq_G j$ if for all $U \subseteq P_G \backslash \{i, j\}$, $U \cup i \in W_G \implies U \cup j \in W_G$. We say that j is more desirable than i.

- This is a partial ordering.
- Say that a game is complete if "$\preceq$" is a total ordering.
- G weighted $\implies$ G complete.

Definition

A simple game G is swap robust if for any two winning coalitions in that game, say S and T, if we swap one player in S with one player in T, then the resulting two coalitions are not both losing.

Definition

A simple game G is swap robust if for any two winning coalitions in that game, say S and T, if we swap one player in S with one player in T, then the resulting two coalitions are not both losing.

Definition

A simple game G is trade robust if for any set $\mathcal{S}$ of winning coalitions in that game, any redistribution of players among the coalitions in $\mathcal{S}$ does not result in all coalitions in $\mathcal{S}$ becoming losing.

Complete Games and Weighted Voting Games

Definition

A simple game G is swap robust if for any two winning coalitions in that game, say S and T, if we swap one player in S with one player in T, then the resulting two coalitions are not both losing.

Definition

A simple game G is trade robust if for any set $\mathcal{S}$ of winning coalitions in that game, any redistribution of players among the coalitions in $\mathcal{S}$ does not result in all coalitions in $\mathcal{S}$ becoming losing.

- G swap robust $\Leftrightarrow$ G complete
- G trade robust $\Leftrightarrow$ G weighted

Composition of Complete Games

Theorem

Let G and H be complete games with more than one distinct minimal winning coalition and no dummy players. Then the composition $C = G \circ_g H$ is complete if and only if g is a member of the weakest desirability class of G.

Composition of Complete Games

Theorem

Let G and H be complete games with more than one distinct minimal winning coalition and no dummy players. Then the composition $C = G \circ_g H$ is complete if and only if g is a member of the weakest desirability class of G.

Proof.

$\Leftarrow$: Let $W_1, W_2 \in W_C$. Write $W_1 = X_1 \cup Y_1$ and $W_2 = X_2 \cup Y_2$. $X_i \cup \{g\}$ is winning in G and Y_i is winning in H if X_i is not winning in G. Three ways to swap a player from W_1 with a player from W_2:

1. $x_1 \in X_1$ with $x_2 \in X_2$: W_1 or W_2 still winning by completeness of G.
2. $y_1 \in Y_1$ with $y_2 \in Y_2$: W_1 or W_2 still winning by completeness of H.
3. $x_1 \in X_1$ with $y_2 \in Y_2$ or vice versa : then $X_2 \cup \{x_1\}$ is winning.

□

Decomposition of Complete Games

- Say that a game G is reducible if there exist G_1, G_2 such that $\min\{|P_{G_1}|, |P_{G_2}|\} > 1$ such that $G = G_1 \circ_g G_2$ for some $g \in G_1$.

Decomposition of Complete Games

- Say that a game G is reducible if there exist G_1, G_2 such that $\min\{|P_{G_1}|, |P_{G_2}|\} > 1$ such that $G = G_1 \circ_g G_2$ for some $g \in G_1$.

Theorem

The set of all complete games with the operation of composition forms a semigroup. Every complete game can be uniquely decomposed (up to isomorphism) as a composition $G = G_1 \circ_{g_1} G_2 \ldots \circ_{g_{n-1}} G_n$ where each G_i is irreducible.

Composition of Weighted Voting Games

- Goal: Given weighted voting games G and H, and $g \in G$, under what conditions is the composition weighted?

Composition of Weighted Voting Games

- Goal: Given weighted voting games G and H, and $g \in G$, under what conditions is the composition weighted?
- If $G \circ_g H$ is weighted, then g must be (one of) the least desirable player in G, or else $G \circ_g H$ is not even complete.

Example

Example

Let $G = [7; 3, 3, 2, 2, 2, 2]$ and let $H = [2; 1, 1, 1]$. Label the two players of weight 3 in G as type A players, the players of weight 2 in G as type B players and the players in H as type C players. We have the following certificate of incompleteness for $G \circ_B H$:

$$(AB^2, ABC^2; A^2C, B^3C)$$

So substituting via the least desirable player is not enough to ensure weightedness of the composition.

A Partial Condition

- If G is weighted then we can always find integer weights and quota for G.

A Partial Condition

- If G is weighted then we can always find integer weights and quota for G.

Theorem

Let G and H be weighted and suppose that there exists an integer system of weights for G such that the least desirable player, g, has weight 1. Then $G \circ_g H$ is weighted.

A Partial Condition

- If G is weighted then we can always find integer weights and quota for G.

Theorem

Let G and H be weighted and suppose that there exists an integer system of weights for G such that the least desirable player, g, has weight 1. Then $G \circ_g H$ is weighted.

- We can prove the theorem by constructing a system of weights for the composition.

Homogeneous Games

Definition (Homogeneous Simple Game)

A homogeneous simple game G is a weighted voting game where it is possible to find a system of weights such that every minimal winning coalition has the same weight.

Homogeneous Games

Definition (Homogeneous Simple Game)

A homogeneous simple game G is a weighted voting game where it is possible to find a system of weights such that every minimal winning coalition has the same weight.

- Ostmann (1984) proved that all homogeneous games can be represented by an integer system of weights with some player having weight 1. Thus, if G is homogeneous and H is weighted, then $G \circ_g H$ is weighted.

Open Questions

- Fully characterise conditions for $G \circ_g H$ to be weighted.

Open Questions

- Fully characterise conditions for $G \circ_g H$ to be weighted.
- Investigate decompositions of arbitrary games.

Open Questions

- Fully characterise conditions for $G \circ_g H$ to be weighted.
- Investigate decompositions of arbitrary games.
- Closure of other classes of game under comppposition. Eg. Is the composition of two homogeneous games in turn homogeneous?