Continuous, nonlinear context-dependence

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Why context-dependence? Prospect theory [KT79].

"An individual's attitude towards money, say, could be described by a book, where each page presents the value function for changes at a particular asset position. Clearly, the value functions described on different pages are not identical: they are likely to become more linear with increases in assets."

Translation: 1) the reference point is context and preferences vary across contexts; 2) dependence on context is typically "*nonlinear*".

What is context?

In psychology a context effect is the influence of environmental factors on the perception of a stimulus.

In economics, in addition to Prospect theory:

- a socio-economic status in a discrete choice model [McF04]?
- a belief of a player in a game [GS03a]?
- a memory in case-based decision theory [GS03b]?
- a set of available alternatives [PX12]? a status quo [MO05] ?

* a "connectome" of a person at decision time [Seu12]?

Perhaps: "Anything that affects preferences, can change and lies beyond the immediate control of the agent."

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Context-free preferences and standard utility

Recall some basic facts about context-free utility representations:

- Let A denote a nonempty set of alternatives.
- Context-free preferences are described by a *single* preference relation $\succ$ on A. Note that $\succ$ is a subset of $A \times A$.
- A function $\mathcal{U} : A \to \mathcal{R}$ is said to *represent* preferences $\succ$ if for any elements a and b,

$$\mathrm{a} \succ b \quad \Leftrightarrow \quad \mathcal{U}(a) > \mathcal{U}(b) \ .$$

State-dependence: as above with "a" being a vector of lotteries.

Context(-dependent) preferences and context utility

Let X denote a nonempty set of contexts.

For each x in X, let $\succ_x$ be a preference relation on alternatives A.

Thus, *context preferences* are described by a family of preference relations $\{\succ_x : x \in X\}$; equivalently by $\{(A, \succ_x) : x \in X\}$.

A function $\mathcal{U} : A \times X \to \mathcal{R}$ is said to represent context preferences $\{(A, \succ_x) : x \in X\}$ if for all a, b in A *and x in X*

$$a \succ_x b \quad \Leftrightarrow \quad \mathcal{U}(a, x) > \mathcal{U}(b, x) \ .$$

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Example of *linear* context-dependence: CBDT

In CBDT, *cases*, are represented by a dimension $1, \ldots, n$, and the context x is understood to be a database (or memory) of cases.

Thus the context space $X := \mathbb{N}^n$ is the set of possible *databases*.

A database x is a vector in $\mathbb{N}^n$ with k^{th} entry equal to the frequency of case k in x.

Each alternative a gives rise to a vector $v(a)$ in $\mathcal{R}^n$ such that for each x in X

$$\mathcal{U}(a, x) = v(a) \cdot x \ .$$

Formal definition of (non)linear context-dependence

Context preferences exhibit *linear context-dependence* when both of the following are true:

1. X can be embedded in a linear space Y;
2. there is a function $A \times Y \rightarrow \mathcal{R}$, that is linear across Y, and whose restriction to $A \times X$ represents context preferences.

Thus, context preferences are said to exhibit *nonlinear context-dependence*, when either of (1) and (2) is false.

Nonlinearity and continuity across contexts

The most important class of nonlinear context preferences are those with a representation that preserves

continuity across contexts.

Note, the main result below also has applications where discontinuity across contexts exists: discontinuous games [BS12]

"When processing sensory input, it is of vital importance for the neural systems to be able to discriminate a novel stimulus from the background of redundant, unimportant signals." [MMB$^+$12]

Formal definition of continuity across contexts (Cac)

Let X be endowed with a *collection* τ of subsets that is closed under finite intersections and arbitrary unions.

(X is then a topological space, and "O in τ" means "O is open".)

Context preferences satisfy (Cac) at x if $a \succ_x b$ implies there exists O in τ such that:

1. x in O; and
2. $a \succ_y b$ for all y in O.

For any $Z \subset X$, context preferences satisfy (Cac) on Z if they are Cac at x for all x in Z.

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Nonlinear, *separately* continuous representation

Theorem (Main result)

Let X be a "perfectly normal" topological space of contexts and let A be discrete and countable. (1) and (2) are equivalent:

1) *preferences $\{(A, \succ_x) : x \in X\}$ satisfy (Asy.), (NT) and are (Cac) on any subset Z of X;*
2) *there exists a function $\mathcal{U} : A \times X \to \mathcal{R}$ that is separately continuous on $A \times Z$ and for all $a, b \in A$, $x \in X$,*

$$a \succ_x b \quad \Leftrightarrow \quad \mathcal{U}(a, x) > \mathcal{U}(b, x) \ .$$

What is a perfectly normal topological space X?

Theorem (Michael's selection theorem [GS00])

The following two statements are equivalent:

1) *X is perfectly normal;*
2) *for any functions $g \leqslant h$ from X to $\mathcal{R}$ that are respectively upper and lower semi-continuous, there is a continuous $f : X \to \mathcal{R}$ such that $g \leqslant f \leqslant h$ and $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$.*

So if the context space is not perfectly normal, then there exist preferences with no utility representation that preserves (Cac).

Jointly continuous representation?

Current examples indicate that the above restriction to separate continuous representations is nonessential. But we will have to consider "metrizable" instead of perfectly normal space. There is the opportunity to build on a large literature in mathematics on separate-to-joint continuity.

The literature [Lev83, BM95, CCM09] on jointly continuous utility representations contains the closest results to the present. Geared for applications in General Equilibrium rather than decision theory. They impose topological structure directly on the product space $\{\succ_x : x \in X\} \subset (A \times A)^X$.

Applications: a foundation for context-dependence

- Wealth/expected wealth in prospect theory [KT79] / [KR06]?
- the set of beliefs of a player in a game [GS03a]?
- the set of memories in case-based decision theory [GS03b]?
- the set of subsets 2^A [PX12]?
- the set A in models of status quo bias [MO05] ?
- the collection of possible connectomes [Seu12]?

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Flood example

Consider a decision-maker devising a “complete, contingent plan” of how to respond to the future threat of a flood.

When the threat becomes imminent:

- two states of nature are of concern: whether or not a *flood* occurs, so $S := \{n, f\}$.
- assume the decision-maker will know the chance, π, of a flood. So contingencies are elements of $\Delta(S) \equiv \Delta \equiv [0, 1]$.
- Given π, the response will be to either *do nothing* or *evacuate*, so $A := \{d, e\}$ is the set of “alternatives/actions”.

Flood example: the Context Preferences approach

- If "flood" is certain, $\pi = 1$ and the planner will evacuate.
- If "no flood" is certain, $\pi = 0$ and the planner does nothing.
- At some point(s) in between, the planner is unsure what to do.

The context preferences approach is close to this line of thinking:

define $\succ_\pi$ on A for each π in Δ

If for some π neither $d \succ_\pi e$ nor $e \succ_\pi d$, then we write $d \sim_\pi e$.

Flood example: conditions on context preferences

Would it not be unusual that both $d \succ_\pi e$ and $e \succ_\pi d$ hold for some π? If we rule this out, context preferences are asymmetric.

Continuity across contexts also seems fairly natural here.

Together these conditions imply that the sets

$$\{\pi \in \Delta : d \succ_\pi e\} \quad \text{and} \quad \{\pi \in \Delta : e \succ_\pi d\}$$

are disjoint and open.

This ensures that the set $\mathcal{N}_{de}$ of π with $d \sim_\pi e$ is nonempty.

Flood example: $\mathcal{N}_{de}$ is a "connected separator"

Given that $d \succ_0 e$ and $e \succ_1 d$, the following would be unusual:

$$e \succ_{\frac{1}{4}} d \quad \text{and} \quad d \prec_{\frac{1}{2}} e\,.$$

A "*betweenness*" condition excludes this sort of preferences: for all

$\pi < \rho < 1$, $\quad d \prec_\pi e \quad$ and $\quad d \prec_1 e \quad$ together imply $\quad d \prec_\rho e$

together with (Asy.) and (C'ty) this is enough to ensure that, *for our flood problem*, $\mathcal{N}_{de}$ is a closed interval.

More generally, to ensure $\mathcal{N}_{de}$ is "*connected*" we also impose "weak Pareto" *across states*. ($a \prec_\pi b \Rightarrow \exists s \in \operatorname{supp}\pi$ with $a \prec_s b$.)

Flood example: conditions for (linear) Expected Utility

The above conditions are *not* enough for an expected utility representation. For an EU representation, $\mathcal{N}_{de}$ must be a point.

If we interpret $\prec_\pi$ as deterministic strict preference, "*thinness*" is an extreme restriction on context preferences:

the planner has to be decisive everywhere except a single point!

Surely, we should allow for preferences such that the decision maker is indecisive on an interval of contexts. If so, we need to use a model that allows for nonlinear context dependence.

Linearity with 3 or more states and alternatives

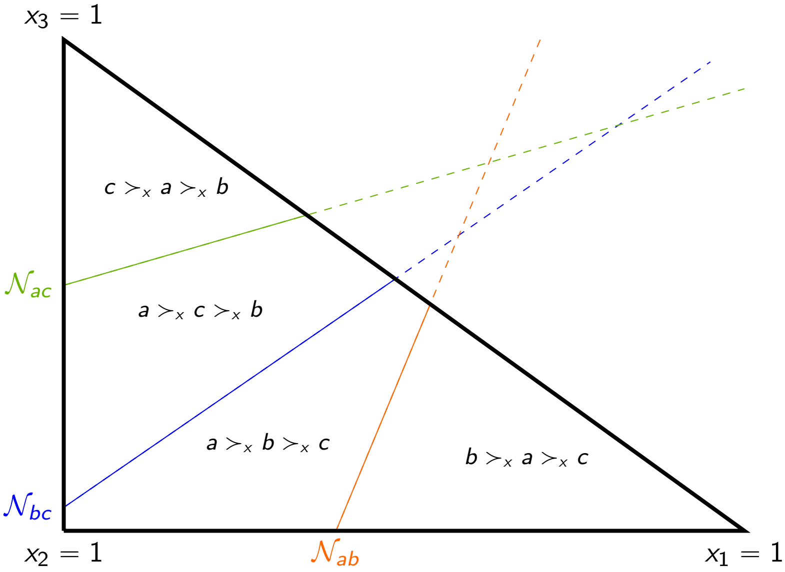
In this setting, the trouble is that we need a "diversity" condition in addition to the "betweenness" and "thinness" discussed above.

3-diversity is the following: for any list (a, b, c) of distinct alternatives there exists an x such that $a \succ_x b \succ_x c$.

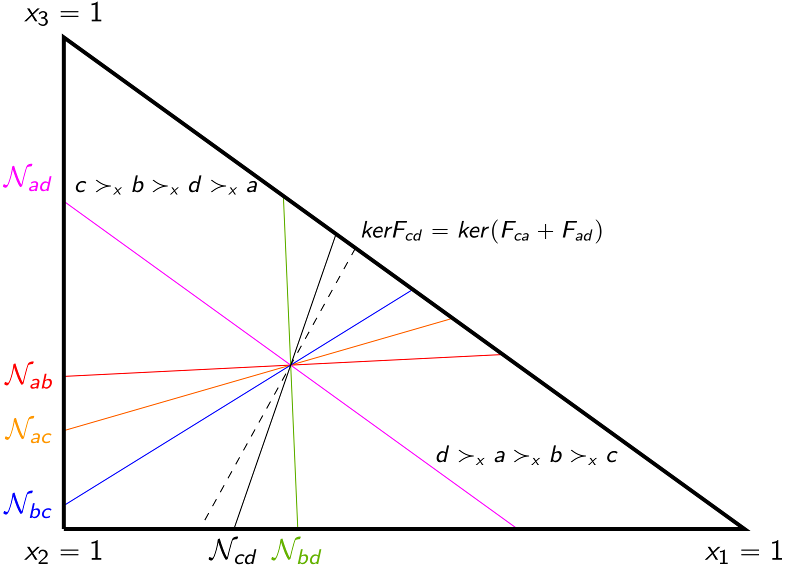
4-diversity is defined similarly (so 24 contexts are needed).

Gilboa-Schmeidler model imposes 4-diversity. It is possible to weaken this to allow for less diverse preferences, but not by much. Either giving up utility or giving up linearity is the only way around the issue. Here we have pursued the latter option.

i) *Non*diverse preferences: *no* linear representation ☹



ii) 3-diverse preferences with *no* linear representation ☹



The case were X is a Lexicographically Ordered space

If X is the interval $[0, 1]^2$ and there is a context-free order $\succ^*$ over $A \times X$, suppose that for each a, the ordering over $X \times \{a\}$ is lexicographic.

Then clearly preferences are nonlinear. In addition, we can't hope for a jointly continuous representation. There is no continuous utility representation.

But we may obtain a Nonlinear, separately continuous Context representation. Moreover, at least in the case where the ordering is the same for all a, the context space is a Perfectly Normal space that is not metrizable.

Thanks!

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