Maximin Rational Expectations Equilibrium

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This talk is based on the following two papers:

[1] L. I. de Castro, M. Pesce and N. C. Yannelis, *A new perspective to rational epectations: maximin rational expectations equilibrium*, working paper, February 2012.

[2] A. Bhowmik, J. Cao and N. C. Yannelis, *Aggregate preferred correspondence and the existence of a maximin REE*

A model of a finite economy

Let $I = \{1, ..., n\}$ be the set of agents. Let Ω be the finite set of states of nature, and $\mathscr{F}$ be the σ-algebra on Ω, representing the set of all events, i.e., $\mathscr{F} = 2^{\Omega}$. The commodity space is the n-dimensional Euclidean space $\mathbb{R}^n$, and $\mathbb{R}^n_+$ is the consumption set for all $(i, \omega) \in I \times \Omega$. *A differential information exchange economy* $\mathscr{E}$ is the following collection

$$\mathscr{E} := \left\{ (\Omega, \mathscr{F});\ (\mathscr{F}_i, u_i, e_i, \pi_i)_{i \in I} \right\},$$

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where for all $i \in I$

- $\mathscr{F}_i$ is a partition of Ω, representing the private information of agent i. If ω is the state of nature that is going to be realized, agent i observes $\mathscr{F}_i(\omega)$, the unique element of $\mathscr{F}_i$ containing ω.
- $u_i : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}$ is a random utility function of agent i, representing his (ex post) preferences. We assume that for all $\omega \in \Omega$, $u_i(\omega, \cdot)$ is continuous.

- $e_i : \Omega \to \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ is a random initial endowment. We assume that e_i is $\mathscr{F}_i$-measurable and $\sum_{i \in I} e_i(\omega) \gg \mathbf{0}$ for any $\omega \in \Omega$.
- π_i is a probability on Ω, representing the prior belief of i. We assume that $\pi_i(\omega) > 0$ for all $\omega \in \Omega$.

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- π_i is a probability on Ω, representing the prior belief of i. We assume that $\pi_i(\omega) > 0$ for all $\omega \in \Omega$.

- At the ex-ante stage ($\tau = 0$), only the above description of the economy is a common knowledge.
- At the interim stage $\tau = 1$, agent t only knows that the realized state of nature belongs to the event $\mathscr{F}_i(\omega^*)$, where $\mathscr{F}_i(\omega^*)$ is the unique member of π_i containing the true state of nature ω^* at $\tau = 2$.
- At the ex-post stage ($\tau = 2$), agents execute the trades according to the contract agreed at period $\tau = 1$, and consumption takes place.

Allocation and price

- A function $x : I \times \Omega \to \mathbb{R}^n_+$ is called an *allocation*. An allocation x is called *feasible* if

$$\sum_{i \in I} x(i, \omega) = \sum_{i \in I} e_i(\omega)$$

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- A price is a non-zero $\mathscr{F}$-measurable function $p : \Omega \to \mathbb{R}^n_+$. Let $\sigma(p)$ be the smallest sub-σ-algebra of $\mathscr{F}$ for which p is measurable. We can think $\sigma(p)$ as the information revealed by the price p. Note that $\sigma(p)$ is generated by a partition $\Pi(p)$ of Ω.

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- The σ-algebra

$$\mathscr{G}_i := \mathscr{F}_i \vee \Pi(p)$$

represents the information combined by the private information $\mathscr{F}_i$ of agent i and the information generated by the price p.

Expected utility

Let $\mathscr{G}_i(\omega)$ be the unique member of $\mathscr{G}_i$ containing ω.

For a consumption bundle $x : \Omega \to \mathbb{R}^n_+$, we consider two different types of expected utility.

- The *Bayesian expected utility* of agent i with respect to $\mathscr{G}_i$ at x in state ω is given by

$$v_i(x|\mathscr{G}_i)(\omega) := \sum_{\omega' \in \mathscr{G}_i(\omega)} u_i(\omega', x(i, \omega')) \times \frac{\pi_i(\omega')}{\pi_i(\mathscr{G}_i(\omega))}.$$

- The *maximin expected utility* of agent i with respect to $\mathscr{G}_i$ at x in state ω is given by

$$\underline{u}_i^{REE}(\omega, x) := \min_{\omega' \in \mathscr{G}_i(\omega)} u_i(\omega', x(\omega')),$$

Rational expectations equilibrium

Let x be a feasible allocation and p is a price system.

- The pair (x, p) is called a *Bayesian REE* if for each $i \in I$,
  1. $x(i, \cdot)$ is $\mathscr{G}_i$-measurable;
  2. $\langle x(i,\omega), p(\omega)\rangle \leq \langle e_i(\omega), p(\omega)\rangle$ for all $\omega \in \Omega$;
  3. $x(i,\omega) \in \arg\max_{y \in B_i(\omega, p(\omega))} v_i(y|\mathscr{G}_i)(\omega)$ for all $\omega \in \Omega$,

where $B_i(\omega, p(\omega))$ is defined as

$$\{y \in (\mathbb{R}^n_+)^\Omega \text{ is } \mathscr{G}_i\text{-measurable} : \langle y(\omega), p(\omega)\rangle \leq \langle e_i(\omega), p(\omega)\rangle\}.$$

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where $B_i^{REE}(\omega, p)$ is defined as

$$\left\{y \in (\mathbb{R}^n_+)^\Omega : \langle y(\omega'), p(\omega')\rangle \leq \langle e_i(\omega'), p(\omega')\rangle \text{ for all } \omega' \in \mathscr{G}_i(\omega)\right\}.$$

An example (Kreps, 1977)

$I = \{1,2\}$, two commodities and two equally probable states of nature, i.e., $\Omega = \{\omega_1, \omega_2\}$. The primitives of the economy are:

$$e_1(\cdot) = \left(\left(\frac{3}{2}, \frac{3}{2} \right), \left(\frac{3}{2}, \frac{3}{2} \right) \right), \mathscr{F}_1 = \{\{\omega_1\}, \{\omega_2\}\};$$

$$e_2(\cdot) = \left(\left(\frac{3}{2}, \frac{3}{2} \right), \left(\frac{3}{2}, \frac{3}{2} \right) \right), \mathscr{F}_2 = \{\{\omega_1, \omega_2\}\}.$$

$$u_1(\omega_1, (x,y)) = \ell n x + y, \quad u_1(\omega_2, (x,y)) = 2\ell n x + y$$

$$u_2(\omega_1, (x,y)) = 2\ell n x + y, \quad u_2(\omega_2, (x,y)) = \ell n x + y.$$

A Bayesian REE does not exist in $\mathscr{E}$, but a unique maximin REE exists in $\mathscr{E}$:

$$(x_1(\omega_1), y_1(\omega_1)) = (1,2), \quad (x_1(\omega_2), y_1(\omega_2)) = (2,1),$$

$$(x_2(\omega_1), y_2(\omega_1)) = (2,1), \quad (x_2(\omega_2), y_2(\omega_2)) = (1,2).$$

Existence results

- Kreps (1979) provided an example that shows that a Bayesian REE does not exist in general.

- Radner (1979) and Allen (1981-2) studied conditions on the existence of a Bayesian REE and obtained some generic existence results.

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Theorem (de Castro, Pesce and Yannelis)

There always exists a maximin REE in $\mathscr{E}$.

- Open question: *Can the above theorem be extended to an economy with infinitely many states of nature, continuum of agents, and even to an infinite dimensional commodity space?*

A model of a continuum economy $\mathscr{E}_c$

- The *space of agents* is a finite measure space (T, Σ, μ).
- The *commodity space* is the n-dimensional Euclidean space $\mathbb{R}^n$, and $\mathbb{R}^n_+$ is the *consumption set* for all $(t, \omega) \in T \times \Omega$.
- The *space of state nature* is a complete probability measure space $(\Omega, \mathscr{F}, \nu)$.
- The *(ex-post) preferences* of agents are represented by a utility function $u : T \times \Omega \times \mathbb{R}^n_+ \to \mathbb{R}$ such that $u(\cdot, \cdot, x)$ is jointly measurable and $u(t, \omega, \cdot)$ is monotone, continuous and concave.
- The *initial endowments* of agents are represented by a jointly measurable function $e : T \times \Omega \to \mathbb{R}^n_+$ such that $\int_T e(\cdot, \omega) d\mu \gg 0$.
- The *private information* of each $t \in T$ is represented by the σ-algebra $\mathscr{F}_t$ generated by a partition Π_t of Ω.
- The *prior belief* of each $t \in T$ is a probability measure $\mathbb{Q}_t$ on Ω.

Aggregate preferred correspondence

Let

$$\Delta := \left\{ p \in \mathbb{R}^n_+ : p \gg 0 \text{ and } \sum_{h=1}^{n} p^h = 1 \right\}.$$

The *budget correspondence* $B : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^n_+$ is defined by

$$B(t, \omega, p) := \left\{ x \in \mathbb{R}^n_+ : \langle p, x \rangle \leq \langle p, e(t, \omega) \rangle \right\}$$

for all $(t, \omega, p) \in T \times \Omega \times \Delta$.

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for all $(t, \omega, p) \in T \times \Omega \times \Delta$.

- Define $C : T \times \Omega \times \Delta \rightrightarrows \mathbb{R}^n_+$ by

$$C(t, \omega, p) := \left\{ y \in \mathbb{R}^n_+ : u(t, \omega, y) \geq u(t, \omega, x), \forall x \in B(t, \omega, p) \right\}.$$

Since $u(t, \omega, \cdot)$ is continuous, $C(t, \omega, p) \neq \emptyset$.

For any $(t, \omega, p) \in T \times \Omega \times \Delta$, let

$$\delta(p) := \min \left\{ p^h : 1 \leq h \leq n \right\}, \text{ and } \gamma(t, \omega, p) := \frac{1}{\delta(p)} \sum_{h=1}^{n} e^h(t, \omega).$$

The *preferred set of agent t* at the price p and state ω is defined as

$$C^X(t,\omega,p) := \{x \in C(t,\omega,p) : x \leq \gamma(t,\omega,p)\mathbf{1}\},$$

and the *aggregate preferred correspondence* is defined by

$$\int_T C^X(\cdot,\cdot,\cdot)d\mu : \Omega \times \Delta \rightrightarrows \mathbb{R}^n_+.$$

Properties of the APC

1. *The APC is non-empty compact-valued.*
2. *For each $\omega \in \Omega$, $\int_T C^X(\cdot,\omega,\cdot)d\mu : \Delta \to \mathbb{R}^n_+$ is Hausdorff continuous.*
3. *For each $p \in \Delta$, $\int_T C^X(\cdot,\cdot,p)d\mu : (\Omega,\mathscr{F},\nu) \rightrightarrows \mathbb{R}^n_+$ is measurable.*

Measurable correspondences

A correspondence $F : (T, \Sigma, \mu) \rightrightarrows (Y, d)$ is *measurable* if

$$F^{-1}(V) := \{t \in T : F(t) \cap V \neq \emptyset\} \in \Sigma.$$

for every open subset $V \subseteq Y$. A measurable $f : (T, \Sigma, \mu) \to (Y, d)$ is called a *measurable selection* of F if $f(t) \in F(t)$ for all $t \in T$.

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Theorem (Characterizations)

Consider the following statements for $F : (T, \Sigma, \mu) \rightrightarrows (Y, d)$:

(1) $F^{-1}(C) \in \Sigma$ *for each closed set* $V \subseteq Y$.

(2) F *is measurable.*

(3) *The function* $t \mapsto d(y, F(t))$ *is* Σ*-measurable for each* $y \in Y$.

(4) $Gr(F) := \{(t, y) \in T \times Y : t \in T, y \in F(t)\} \in \Sigma \otimes \mathscr{B}(Y)$.

Then $(1) \Rightarrow (2) \Rightarrow (3)$. *If* F *is closed-valued and* Y *is separable, then all of these statements are equivalent.*

Measurable selections

Kuratowski-Ryll-Nardzewski Selection Theorem (1965)

If $F : (T, \Sigma, \mu) \rightrightarrows (Y, d)$ is a closed-valued and measurable correspondence into a complete separable metric space, then F admits a measurable selection.

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von Neumann (1949)-Aumann (1969) Selection Theorem

If $F : (T, \Sigma, \mu) \rightrightarrows (Y, d)$ is a correspondence into a complete separable metric space such that $Gr(F) \in \Sigma \otimes \mathscr{B}(Y)$, then F admits a measurable almost everywhere selection.

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A Hausdorff topological is called a *Suslin space* if it is a continuous image of some Polish space.

Sainte-Beuve Selection Theorem (1974)

If $F : (T, \Sigma, \mu) \rightrightarrows Y$ is a correspondence into a Suslin space such that $Gr(F) \in \Sigma \otimes \mathscr{B}(Y)$, then F admits a measurable selection.

A general existence result

Theorem (Bhowmik, C. and Yannelis)

There always exists a maximin REE in $\mathscr{E}_c$.

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Proof. Consider the correspondence $Z : \Omega \times \Delta \rightrightarrows \mathbb{R}^n$, defined by

$$Z(\omega, p) := \int_T C^X(\cdot, \omega, p) d\mu - \int_T e(\cdot, \omega) d\mu.$$

Then, Z is non-empty compact-valued and jointly measurable. By the existence theorem of a Walrasian equilibrium due to Hildenbrand in 1974, the correspondence $F : \Omega \rightrightarrows \Delta$, defined by

$$F(\omega) := \{p \in \Delta : Z(\omega, p) \cap \{0\} \neq \emptyset\},$$

is non-empty valued. Since $Gr(F) = Z^{-1}(\{0\})$ and Z is jointly measurable, F has a measurable graph.

Proof continued

Now, the Sainte-Beuve selection theorem implies that F admits a measurable selection $\hat{p} : \Omega \to \Delta$.

Proof continued

Now, the Sainte-Beuve selection theorem implies that F admits a measurable selection $\hat{p} : \Omega \to \Delta$.

By the definition of Z, there exists a feasible allocation x such that $x \in C^X(t, \omega, \hat{p}(\omega))$ for almost all $t \in T$ and all $\omega \in \Omega$. Thus, $x(t, \omega) \in B_t(\omega, \hat{p}(\omega))$ for almost all $t \in T$ and all $\omega \in \Omega$. Define

$$T_\omega := \{t \in T : x(t, \omega) \in B_t(\omega, \hat{p}(\omega)) \cap C(t, \omega, \hat{p}(\omega))\}.$$

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$$T_\omega := \{t \in T : x(t, \omega) \in B_t(\omega, \hat{p}(\omega)) \cap C(t, \omega, \hat{p}(\omega))\}.$$

Finally, define a function $\hat{x} : T \times \Omega \to \mathbb{R}^n_+$ such that if $t \in T_\omega$, $\hat{x}(t, \omega) = x(t, \omega)$, and if $t \in T \setminus T_\omega$, $\hat{x}(t, \omega)$ is any point in $B_t(\omega, \hat{p}(\omega)) \cap C(t, \omega, \hat{p}(\omega))$.

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It can be verified that $(\hat{x}, \hat{p})$ is a maximin rational expectation equilibrium in $\mathscr{E}_c$. $\square$

Other properties of a maximin REE in $\mathscr{E}$

- There always exists a MREE which satisfies the budget set with an equality.
- Under certain assumptions the equilibrium price is strictly positive in each state of nature.
- If the utility functions are private information measurable, then for each agent $i \in I$, the maximin utility at any maximin REE allocation is constant in each event of the partition $\mathscr{G}_i$.
- If the utility functions are private information measurable and monotone, then any maximin REE allocation is maximin efficient.
- Any maximin REE allocation is maximin coalitional incentive compatible.

The end

Thank You $\cdots$