Hierarchical simple games, Structure and Characterisations

Ali Hameed and Arkadii Slinko

Department of Mathematics
The University of Auckland

21 Feb 2012

Secret Sharing Schemes (SSS's)

Secret Sharing Schemes (SSS's)

Definition

A *SSS* is a method of distributing shares to a number of participants, so that a subset of participants can determine the secret if and only if that subset is authorised to do so.

Authorised/Winning coalitions

Authorised/Winning coalitions

- The collection of authorised subsets is called *access structure*.

Authorised/Winning coalitions

- The collection of authorised subsets is called *access structure*.
- From Game Theory perspective, an access structure is a *simple game*, and authorised subsets are *winning coalitions*.

Authorised/Winning coalitions

- The collection of authorised subsets is called *access structure*.
- From Game Theory perspective, an access structure is a *simple game*, and authorised subsets are *winning coalitions*.

Secret $S = 10$

Secret $S = 10$

Player A

Player B

Secret $S = 10$

Player A
Share $S_1 = 6$

Player B
Share $S_2 = 4$

Secret $S = 10$

Player A
Share $S_1 = 6$

Player B
Share $S_2 = 4$

$S_1 + S_2 = 10$

Secret $S = 10$

Player A
Share $S_1 = 6$

Player B
Share $S_2 = 4$

$S_1 + S_2 = 10$

A SSS is called *perfect* if unauthorised coalitions receive zero information about the secret.

Secret $S < ab$

Player A

Player B

Secret $S < ab$

Player A
$S_1 = S \bmod a$

Player B
$S_2 = S \bmod b$

Secret $S < ab$

Player A
$S_1 = S \bmod a$

Player B
$S_2 = S \bmod b$

Solve using the Chinese Remainder Theorem

Secret $S < ab$

Player A
$S_1 = S \bmod a$

Player B
$S_2 = S \bmod b$

Solve using the Chinese Remainder Theorem

And this scheme is not perfect

- *Length* of a share or secret is the number of bits used to write it.

- *Length* of a share or secret is the number of bits used to write it.

Definition

A perfect SSS is called *ideal* if the length of the share is equal to the length of the secret.

- *Length* of a share or secret is the number of bits used to write it.

Definition

A perfect SSS is called *ideal* if the length of the share is equal to the length of the secret.

Examples of ideal SSS are numerous, in particular, hierarchical SSS that will be considered later are ideal.

The Big Project

The Big Project

Problem

Characterise all ideal simple games.

The Big Project

Problem

Characterise all ideal simple games.

- This problem turned out to be extremely difficult, so the focus shifted to some subclasses of ideal simple games.

The Big Project

Problem

Characterise all ideal simple games.

- This problem turned out to be extremely difficult, so the focus shifted to some subclasses of ideal simple games.
- The first subclass is called Weighted Simple Games (WSG), introduced by [von Neumann and Morgenstern, 1944].

The Big Project

Problem

Characterise all ideal simple games.

- This problem turned out to be extremely difficult, so the focus shifted to some subclasses of ideal simple games.
- The first subclass is called Weighted Simple Games (WSG), introduced by [von Neumann and Morgenstern, 1944].

Definition (Weighted simple games)

Let $w_1, \ldots, w_n$ be a system of non-negative weights and $q \geq 0$. We define

$$\Gamma = \{X \in 2^U \mid \sum_{i \in X} w_i \geq q\}.$$

Example of weightedness

Example

An electronic fund transfer of a large sum of money can be authorised by either:

1. two general managers, or
2. three senior tellers, or
3. one general manager and two senior tellers.

If the two general managers have weights $w_{1a} = w_{1b} = 3$, and If the three senior tellers have weights $w_{2a} = w_{2b} = w_{2c} = 2$,

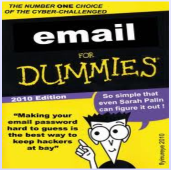
such that $q = 6$, then this is a weighted game.

Characterisation of Ideal WSG's

- Ideal WSG's have been characterised.

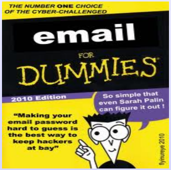
Characterisation of Ideal WSG's

- Ideal WSG's have been characterised.
- A player whose removal from a winning coalition keeps the colaition winning is *Dummy*.



Characterisation of Ideal WSG's

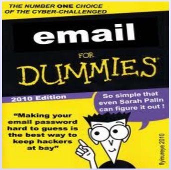
- Ideal WSG's have been characterised.
- A player whose removal from a winning coalition keeps the colaition winning is *Dummy*.



- The characterisation is for simple games with no dummies.

Characterisation of Ideal WSG's

- Ideal WSG's have been characterised.
- A player whose removal from a winning coalition keeps the colaition winning is *Dummy*.



- The characterisation is for simple games with no dummies.

Theorem (Beimel, Tassa and Weinreb, 2008)

A WSG Γ is ideal iff one of the following three conditions holds:

- Γ *is a hierarchical simple game of at most two levels;*
- Γ *is a tripartite simple game;*
- Γ *is a composition of two ideal WSG's.*

The next step

Definition (Roughly Weighted (RWSG))

- If $X \in 2^U$ is such that $\sum_{i \in X} w_i > q$, then X is authorized (belongs to Γ);

The next step

Definition (Roughly Weighted (RWSG))

- If $X \in 2^U$ is such that $\sum_{i \in X} w_i > q$, then X is authorized (belongs to Γ);
- If $Y \in 2^U$ is such that $\sum_{i \in Y} w_i < q$, then Y is not authorized.

The next step

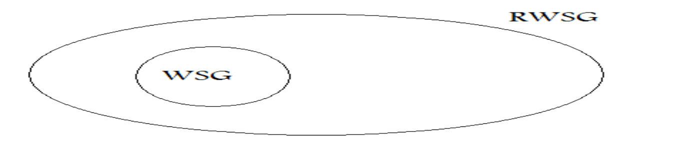
Definition (Roughly Weighted (RWSG))

- If $X \in 2^U$ is such that $\sum_{i \in X} w_i > q$, then X is authorized (belongs to Γ);
- If $Y \in 2^U$ is such that $\sum_{i \in Y} w_i < q$, then Y is not authorized.
- If $Z \in 2^U$ is such that $\sum_{i \in Z} w_i = q$, then a tie-breaking rule will decide whether the set is authorized or not.

The next step

Definition (Roughly Weighted (RWSG))

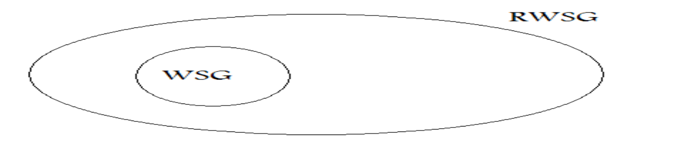
- If $X \in 2^U$ is such that $\sum_{i \in X} w_i > q$, then X is authorized (belongs to Γ);
- If $Y \in 2^U$ is such that $\sum_{i \in Y} w_i < q$, then Y is not authorized.
- If $Z \in 2^U$ is such that $\sum_{i \in Z} w_i = q$, then a tie-breaking rule will decide whether the set is authorized or not.



The next step

Definition (Roughly Weighted (RWSG))

- If $X \in 2^U$ is such that $\sum_{i \in X} w_i > q$, then X is authorized (belongs to Γ);
- If $Y \in 2^U$ is such that $\sum_{i \in Y} w_i < q$, then Y is not authorized.
- If $Z \in 2^U$ is such that $\sum_{i \in Z} w_i = q$, then a tie-breaking rule will decide whether the set is authorized or not.



Problem

Characterise all ideal RWSGs.

The natural class of Hierarchical simple games (HSG's)

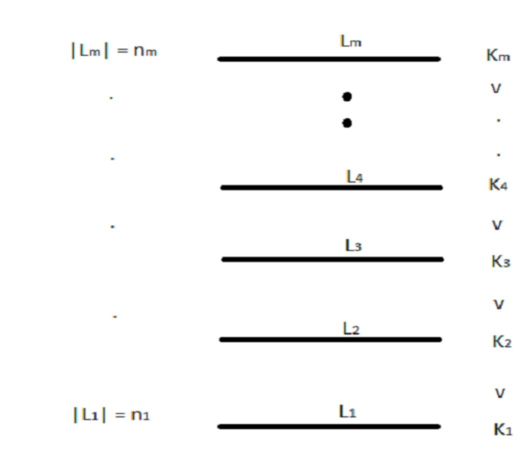


Figure: An m-level hierarchical simple game

Disjunctive hierarchical simple game (DHSG)

Definition

In a DHSG, a coalition of participants is authorised if it contains at least k_1 members from level 1, *or* k_2 members from levels 1 and 2, *or* k_3 members from levels 1, 2 and 3 etc.

Disjunctive hierarchical simple game (DHSG)

Definition

In a DHSG, a coalition of participants is authorised if it contains at least k_1 members from level 1, *or* k_2 members from levels 1 and 2, *or* k_3 members from levels 1, 2 and 3 etc.

Example

An electronic fund transfer of a large sum of money can be authorised by either:

1. two general managers, or
2. three senior tellers, or
3. one general manager and two senior tellers.

So it is two levels:
The general managers L_1 with $k_1 = 2$, and
The senior tellers L_2 with $k_2 = 3$.

Conjunctive Hierarchical simple game (CHSG)

Definition

In a CHSG, a coalition of participants is authorised if it contains at least k_1 members from level 1, *and* k_2 members from levels 1 and 2, *and* k_3 members from levels 1, 2 and 3 etc.

Conjunctive Hierarchical simple game (CHSG)

Definition

In a CHSG, a coalition of participants is authorised if it contains at least k_1 members from level 1, *and* k_2 members from levels 1 and 2, *and* k_3 members from levels 1, 2 and 3 etc.

Example

A passage of a resolution in the United Nations Security Council requires the vote of at least:

1. 9 members in total, and
2. at least 5 permanent members.

So it is two levels:
The permenant members L_1 with $k_1 = 5$, and
The non-permanent members L_2 with $k_2 = 9$.

- A coalition X is said to be *blocking*, if X^c is losing.

- A coalition X is said to be *blocking*, if X^c is losing.

Definition

The simple games G and G^d are duals of each other if the winning coalitions of G^d are the blocking coalitions of G.

- A coalition X is said to be *blocking*, if X^c is losing.

Definition

The simple games G and G^d are duals of each other if the winning coalitions of G^d are the blocking coalitions of G.

Theorem

DHSG's and CHSG's are duals of each other.

- A coalition X is said to be *blocking*, if X^c is losing.

Definition

The simple games G and G^d are duals of each other if the winning coalitions of G^d are the blocking coalitions of G.

Theorem

DHSG's and CHSG's are duals of each other.

Example

The DHSG $\mathbf{k} = (2, 4), \mathbf{n} = (2, 4)$, is the dual game of the CHSG $\mathbf{k} = (1, 3), \mathbf{n} = (2, 4)$

- A coalition X is said to be *blocking*, if X^c is losing.

Definition

The simple games G and G^d are duals of each other if the winning coalitions of G^d are the blocking coalitions of G.

Theorem

DHSG's and CHSG's are duals of each other.

Example

The DHSG $\mathbf{k} = (2, 4), \mathbf{n} = (2, 4)$, is the dual game of the CHSG $\mathbf{k} = (1, 3), \mathbf{n} = (2, 4)$
$\{1^2, 2\}$ is blocking in DHSG and is winning in CHSG,
$\{1, 2^2\}$ is blocking in DHSG and is winning in CHSG.

Weighted DHSG's

Weighted DHSG's

A *trivial level* is a level whose players either form authorised coalitions individually, or they are dummies.

Weighted DHSG's

A *trivial level* is a level whose players either form authorised coalitions individually, or they are dummies.

Theorem (Beimel, Tassa and Weinreb, 2008)

A HSG is weighted iff it has up to four levels but only two non-trivial. The non-trivial levels L_i, L_{i+1} must have either:

- $k_{i+1} = k_i + 1$, *or*
- $n_{i+1} = k_{i+1} - k_i + 1$.

Weighted DHSG's

A *trivial level* is a level whose players either form authorised coalitions individually, or they are dummies.

Theorem (Beimel, Tassa and Weinreb, 2008)

A HSG is weighted iff it has up to four levels but only two non-trivial. The non-trivial levels L_i, L_{i+1} must have either:

- $k_{i+1} = k_i + 1$, *or*
- $n_{i+1} = k_{i+1} - k_i + 1$.

- An analogue of the above theorem for CHSG's is found by Duality.

Some definitions

Some definitions

- In an access structure Γ, i is said to be more senior than j, formally $i \succeq_\Gamma j$, if $X \cup \{j\} \in \Gamma$ implies $X \cup \{i\} \in \Gamma$ for every set $X \subseteq U$ not containing i and j. The game is called *complete* if $\succeq_\Gamma$ is a total order.

Some definitions

- In an access structure Γ, i is said to be more senior than j, formally $i \succeq_\Gamma j$, if $X \cup \{j\} \in \Gamma$ implies $X \cup \{i\} \in \Gamma$ for every set $X \subseteq U$ not containing i and j. The game is called *complete* if $\succeq_\Gamma$ is a total order.
- A *shift* is a replacement of a player by a less senior one.

Some definitions

- In an access structure Γ, i is said to be more senior than j, formally $i \succeq_\Gamma j$, if $X \cup \{j\} \in \Gamma$ implies $X \cup \{i\} \in \Gamma$ for every set $X \subseteq U$ not containing i and j. The game is called *complete* if $\succeq_\Gamma$ is a total order.
- A *shift* is a replacement of a player by a less senior one.

- A *shift-maximal* coalition is a losing coalition whose every superset is winning and cannot be obtained from any losing coalition by a shift.

Which HSGs are roughly weighted but not weighted?

The main tool in the characterisation:

Theorem

The class of disjunctive hierarchical simple games are exactly the class of complete games with a unique shift-maximal losing coalition.

Which HSGs are roughly weighted but not weighted?

The main tool in the characterisation:

Theorem

The class of disjunctive hierarchical simple games are exactly the class of complete games with a unique shift-maximal losing coalition.

- An analogue of the above theorem for CHSG's is also found by Duality.

Roughly Weighted DHSG that are non-weighted

Theorem

A DHSG is roughly weighted if and only if it has up to three non-trivial levels, such that:

Roughly Weighted DHSG that are non-weighted

Theorem

A DHSG is roughly weighted if and only if it has up to three non-trivial levels, such that:

- *If it has two levels* L_i, L_{i+1}*, then* $k_{i+1} = k_i + 2$*;*

Roughly Weighted DHSG that are non-weighted

Theorem

A DHSG is roughly weighted if and only if it has up to three non-trivial levels, such that:

- *If it has two levels L_i, L_{i+1}, then $k_{i+1} = k_i + 2$;*
- *If it has three levels, then some restrictions apply to the number of players of each level;*

Roughly Weighted DHSG that are non-weighted

Theorem

A DHSG is roughly weighted if and only if it has up to three non-trivial levels, such that:

- *If it has two levels L_i, L_{i+1}, then $k_{i+1} = k_i + 2$;*
- *If it has three levels, then some restrictions apply to the number of players of each level;*

Example

(i) $k_1 = 2, k_2 = 3, k_3 = 4$, with $n_1 = 2, n_2 = 2, n_3 = 3$ denoted $k = (2, 3, 4), n = (2, 2, 3)$;

Roughly Weighted DHSG that are non-weighted

Theorem

A DHSG is roughly weighted if and only if it has up to three non-trivial levels, such that:

- *If it has two levels* L_i, L_{i+1}*, then* $k_{i+1} = k_i + 2$*;*
- *If it has three levels, then some restrictions apply to the number of players of each level;*

Example

(i) $k_1 = 2, k_2 = 3, k_3 = 4$, with $n_1 = 2, n_2 = 2, n_3 = 3$ denoted $k = (2, 3, 4), n = (2, 2, 3)$;
(ii) $k = (3, 4, 6), n = (3, 3, 3)$.

Roughly Weighted DHSG that are non-weighted

Theorem

A DHSG is roughly weighted if and only if it has up to three non-trivial levels, such that:

- *If it has two levels L_i, L_{i+1}, then $k_{i+1} = k_i + 2$;*
- *If it has three levels, then some restrictions apply to the number of players of each level;*

Example

(i) $k_1 = 2, k_2 = 3, k_3 = 4$, with $n_1 = 2, n_2 = 2, n_3 = 3$ denoted $k = (2, 3, 4), n = (2, 2, 3)$;
(ii) $k = (3, 4, 6), n = (3, 3, 3)$.

- A characterisation for the Roughly weighted CHSG's is also found by Duality.

THANK YOU !