Best reply dynamics for scoring rules

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- Player (voter) action is to submit an expressed vote (possibly different from its sincere preference).
- We are interested in the equilibrium result to predict the winner of election.
- Gibbard-Satterthwaite and other theorems show that dominant strategies don't always exist.
- The common solution concept is Nash Equilibrium (NE).
- Far too many NE exist and some of them are trivial. Also, voters cannot coordinate on one equilibrium.

Dynamic process of convergence

- We can use learning models or dynamic process (iterative games) as a coordination device
- Fudenberg and Levine (1998): "In some cases, most learning models do not converge to any equilibrium and just coincide with the notion of rationalizability."
- However, if it converges, it necessarily finds a NE.
- It has application for voting in finding consensus: doodle.com, etc.

Best reply dynamics

- In the voting case it has only been used by Meir et al, for best-reply dynamics (BRD) as far as we know (AAAI 2010).
- The first player moves, then another one responds, etc
- Players move one at a time and in each move, they do a best reply.
- Myopic moves, no communication between players and zero knowledge of others.

Basic setup

- We have a set $\mathcal{C}$ of alternatives (candidates) and set $\mathcal{V}$ of voters, with $m := |\mathcal{C}|, n := |\mathcal{V}|$.
- Each voter v submits a permutation $L(v)$ of the candidates. This defines the set $\mathcal{T}$ of types, and $|\mathcal{T}| = m!$.
- A profile is a function $\mathcal{V} \to \mathcal{T}$. A voting situation is a multiset from $\mathcal{T}$ with total weight n.
- The scoring rule determined by a vector w with $w_1 \geq w_2 \geq \cdots \geq w_{m-1} \geq w_m$ assigns the score

$$|c| := \sum_{t \in T} |\{v \in \mathcal{V} \mid L(v) = t\}| w_{L(v)^{-1}(c)}.$$

- Special cases:
  - plurality: $w = (1, 0, 0, \ldots, 0)$;
  - antiplurality (veto): $w = (1, 1, \ldots, 1, 0)$;
  - Borda: $w = (m - 1, m - 2, \ldots, 1, 0)$.

An example of BRD

Consider antiplurality system with 2 voters $\mathcal{V} = \{1, 2\}$ and 4 candidates $\mathcal{C} = \{a, b, c, d\}$, alphabetically tie-breaking. The sincere profile is $P_0 = (acbd, bacd)$. If voters start from sincere state, we have

$(-d, -d)\{a\} \xrightarrow{2} (-d, -a)\{b\} \xrightarrow{1} (-b, -a)\{c\} \xrightarrow{2}$
$(-b, -c)\{a\}$

The best reply is not unique. For example, the last move by second player can instead be $-d$. However, $-c$ (vetoing the current winner) is what we call Restricted Best Reply (RBR) for antiplurality .

Summary of Meir et al

- Plurality voting
- Other assumptions:
  - Behaviour : RBR at each step or best reply.
  - Indifference: keep last move or truth-biased.
  - Initial state: sincere profile or arbitrary profile.
  - Tiebreaking: deterministic or uniform random.
  - Voters: unweighted or weighted.
- Convergence for plurality under red hypotheses in at most m^2n^2 steps.

Summary of Meir et al

Results:

- Deterministic tiebreaking from an arbitrary initial state converges for unweighted voters.
- The winner is the sincere winner or a candidate at most 1 point behind initially.
- In each case, changing each red hypothesis and keeping the others yields examples of non-convergence.

Example of cycle in BRD

Example

Consider the sincere profile $P_0 = (abc, bca)$ and voting rule Borda and alphabetical tie-breaking.

$(abc, bca)\{b\} \xrightarrow{1} (acb, bca)\{a\} \xrightarrow{2} (acb, cba)\{c\} \xrightarrow{1}$
$(abc, cba)\{a\} \xrightarrow{2} (abc, bca)\{b\} \ \Diamond$.

Restricted Best Reply Dynamics for antiplurality

- Best reply is not unique, because several preference orders may yield the same result.
- Is there a natural restricted best reply which is unique?
- One answer can be the best reply that maximizes the winning score margin of the new winner.

Restricted Best Reply Dynamics for antiplurality

- Without loss of generality, we can assume that $S_i = \{-c|c \in C\}$
- o_t is the winner set after the move of player i at time t and is the current winner set .
- Best reply: $-a \rightarrow -b$ where $b \neq a$.

  Type 1: $a \notin o_t$ and $b \in o_{t-1}$

  Type 2: $a \in o_t$ and $b \notin o_{t-1}$

  Type 3: $a \in o_t$ and $b \in o_{t-1}$
- For plurality and antiplurality, allowing type 2 moves can lead to a cycle. We call type 1 and 3 as **RBR**.
- In randomized tie-breaking, the player vetoes the least preferred member of o_{t-1}.

Definition

(**set of potential winners**) The set of potential winners at time t, W_t, those candidates who have a chance of winning at the next step (time $t+1$), depending on the different RBR of voters.

$$W_t = \{c \mid \text{if a player moves } -c \rightarrow -b \text{ at time } t+1 \text{, then } \; c \in o_{t+1}\}$$

Lemma

If $t < t'$ then $W_t \subseteq W'_t$.

Proof.

- Let $c \in W_{t-1}$, and we have an improvement step $-a \to -b$ at time t. Then we show $c \in W_t$.
- If it is type 3, it is easy to show $-c \to -a$ makes c winner.
- If it is a type 1, let $b' = o_t$. Note that $b' \notin \{a, b\}$. Then we show $-c \to -b'$ makes c winner. It uses the transitivity of lexicographic order which may not be true for an arbitrary deterministic tie-breaking rule.

□

Lemma

Each voter has at most one type 1 move and at most $m-1$ moves of type 3.

Proof.

1. Suppose a step $-a \xrightarrow{i} -b$ is a type 1 move by voter i at time t. With proof by contradiction, we show this improvement step is the first improvement step of voter i.
2. as at every step $-a \xrightarrow{i} -b$ of type 3, it must hold that $a \succ_i b$ because of the definition of improvement step.

□

Conclusion: RBRD for $G(\mathcal{V}, \mathcal{C}, A)$ with alphabetical tie-breaking will converge to a NE from any state in at most mn steps.

Definition

(Stochastic dominance improvement step) Voter v prefers an outcome with winner set W to an outcome with winner set W' is preferred to W' if and only if for each $k = 1 \cdots m$, the probability of electing one of the first k candidates given outcome W should be no less than given W'.

Lemma

If $t < t'$ then $W_t \subseteq W'_t$.

Lemma

Each voter has at most one type 1 move and at most $m-1$ moves of type 3.

Proof.

For type 1 move, similarly proof by contradiction.
For type 3 move $-a \to -b$, we show the probability of winning of a has increased and b has decreased. Therefore, $a \succ_i b$. □

Conclusion: Stochastic dominance RBRD for $G(\mathcal{V}, \mathcal{C}, A)$ with randomized tie-breaking will converge to a NE from any state in at most mn steps.

RBRD for plurality

Type 1: $a \notin o_{t-1}$ and $b \in o_t$

Type 3: $a \in o_{t-1}$ and $b \in o_t$

$W_t = \{c \mid$ **if a player moves** $a \to c$ **and** $a \in o_{t-1}$ **then** $c \in o_t\}$

Lemma

If $t < t'$ then $W_t' \subseteq W_t$.

Lemma

The number of type 1 moves are at most m and each voter has at most $m - 1$ moves of type 3.

Theorem

RBRD $G(V, C, P)$, will converge to a NE from any state in at most $m + (m - 1)n$ steps.

Cycles for more general scoring rules

Cycle for scoring rules close to Plurality:

- Suppose we have 3 candidates a, b and c and $P_0 = (abc, bca)$. The scoring rule is $w = (1, \alpha, 0)$;$\alpha \leq \frac{1}{2}$ and we use alphabetical tie-breaking.
  $(abc, bca)\{b\} \xrightarrow{1} (acb, bca)\{a\} \xrightarrow{2} (acb, cba)\{c\} \xrightarrow{1}$
  $(abc, cba)\{a\} \xrightarrow{2} (abc, bca)\{b\} \;\Diamond$
- general m and $n = 2$
  $(ab\cdots c, bc\cdots a)\{b\} \xrightarrow{1} (a\cdots cb, bc\cdots a)\{a\} \xrightarrow{2}$
  $(a\cdots cb, cb\cdots a)\{c\} \xrightarrow{1} (ab\cdots c, cb\cdots a)\{a\} \xrightarrow{2}$
  $(ab\cdots c, bc\cdots a)\{b\} \;\Diamond$

Cycle for scoring rules close to antiplurality: $m = 3, n = 4$

Suppose we have 3 candidates a, b and c. The sincere profile is $P_0 = (abc, bac, cab, bca)$. Our scoring rule is $(1, \alpha, 0)$; $\frac{1}{2} \leq \alpha < 1$ with alphabetical tie-breaking.
$(abc, bac, cab, bca)\{b\} \xrightarrow{1} (acb, bac, cab, bca)\{a\} \xrightarrow{4}$
$(acb, bac, cab, cba)\{c\} \xrightarrow{1} (abc, bac, cab, cba)\{a\} \xrightarrow{4}$
$(abc, bac, cab, bca)\{b\}$ $\Diamond$

Order of players matters

Consider Borda system with 4 voters and 3 candidates, $P_0 = (acb, acb, cab, cba)$ and alphabetically tie-breaking

$(acb, acb, cab, cba)\{c\} \xrightarrow{1} (abc, acb, cab, cba)\{a\} \xrightarrow{3}$
$(abc, acb, cba, cba)\{c\} \xrightarrow{2} (abc, abc, cba, cba)\{a\} \xrightarrow{4}$
$(abc, abc, cba, bca)\{b\} \xrightarrow{1} (acb, abc, cba, bca)\{1\} \xrightarrow{4}$
$(acb, abc, cba, cba)\{c\} \xrightarrow{1} (abc, abc, cba, bca) \ \lozenge$

P is the same as P after fourth move and the cycle starts.

Now let's consider another order for the players.

$(acb, acb, cba, cab)\{c\} \xrightarrow{1} (abc, acb, cba, cab)\{a\} \xrightarrow{4}$
$(abc, acb, cba, cba)\{c\} \xrightarrow{2} (abc, abc, cba, cba)\{a\} \xrightarrow{3}$
$(abc, abc, bca, cba)\{b\} \xrightarrow{4} (abc, abc, bca, cab)\{a\}$

Conclusion

- The possibility of winning of a candidate depends on the type of improvement step and also candidate's priority in tie-breaking (max $d = 2$)
- The number of type 2 moves are not bounded, so we need to use RBR for convergence.
- We need to use stochastic dominance RBR for randomized tie-breaking.
- The results of convergence do not happen for a non-linear deterministic tie-breaking rule
- The order of players matters in convergence, equilibrium result and also speed of convergence.